EXTENSIONS OF AN AMAZING IDENTITY OF RAMANUJAN

KWANG-WU CHEN

ABSTRACT. To begin with an amazing identity of Ramanujan, we derive an algorithm. Using this algorithm we get a lot of similar identities. Finally, we apply this algorithm to reformulate Ramanujan's 6-10-8 identity.

1. INTRODUCTION

Ramanujan [5] recorded the following amazing identity. If the sequences (a_k), (b_k), and (c_k) are defined by

$$\begin{aligned} \sum_{k\geq 0} a_k x^k &= \frac{1+53x+9x^2}{1-82x-82x^2+x^3}, \\ \sum_{k\geq 0} b_k x^k &= \frac{2-26x-12x^2}{1-82x-82x^2+x^3}, \\ \sum_{k\geq 0} c_k x^k &= \frac{2+8x-10x^2}{1-82x-82x^2+x^3}, \end{aligned} \tag{1.1}$$

then

$$a_k^3 + b_k^3 = c_k^3 + (-1)^k, \qquad \text{for all } k \geq 0. \tag{1.2}$$

Two proofs of this identity and a plausible explanation of how Ramanujan might have discovered it have been given by Hirschhorn [1, 2].

Motivated by Hirschhorn's explanation, McLaughlin [3] gives a similar identity involving eleven sequences.

Example 1.1. *(McLaughlin [3]) Let the sequences of integers* (a_k), (b_k), (c_k), (d_k), (e_k), (f_k), (p_k), (q_k), (r_k), *and* (s_k) *be defined by*

$$\begin{aligned} \sum_{k\geq 0} a_k x^k &= \frac{-3-164x-x^2}{1-99x+99x^2-x^3}, & \sum_{k\geq 0} b_k x^k &= \frac{-1-134x+7x^2}{1-99x+99x^2-x^3}, \\ \sum_{k\geq 0} c_k x^k &= \frac{1-298x+x^2}{1-99x+99x^2-x^3}, & \sum_{k\geq 0} d_k x^k &= \frac{7-228x+5x^2}{1-99x+99x^2-x^3}, \\ \sum_{k\geq 0} e_k x^k &= \frac{5-258x-3x^2}{1-99x+99x^2-x^3}, & \sum_{k\geq 0} f_k x^k &= \frac{3-94x+3x^2}{1-99x+99x^2-x^3}, \\ \sum_{k\geq 0} p_k x^k &= \frac{-3-138x+5x^2}{1-99x+99x^2-x^3}, & \sum_{k\geq 0} q_k x^k &= \frac{-1-244x-3x^2}{1-99x+99x^2-x^3}, \\ \sum_{k\geq 0} r_k x^k &= \frac{7-254x-x^2}{1-99x+99x^2-x^3}, & \sum_{k\geq 0} s_k x^k &= \frac{5-148x+7x^2}{1-99x+99x^2-x^3}. \end{aligned} \tag{1.3}$$

Then for each $k \geq 0$, $n = 1, 2, 3, 4, 5$, we have

$$a_k^n + b_k^n + c_k^n + d_k^n + e_k^n + f_k^n = p_k^n + q_k^n + r_k^n + s_k^n + 3^n + 1. \tag{1.4}$$

According to these good ideas, we present a systematic approach in Section 3. By this algorithm we could give a lot of identities like those of Ramanujan and McLaughlin.

In Section 4, we apply this algorithm to get a different type of identity, Ramanujan's $6-10-8$ identity and Hirschhorn's $3-7-5$ identity.

2. Prelimilaries

Let a sequence (h_k) be defined by

$$h_0 = 0, h_1 = 1, h_{k+2} = ah_{k+1} + bh_k, \quad \text{for all } k \geq 0, \tag{2.1}$$

where a and b are nonzero integers.

The solutions of the characteristic equation $x^2 - ax - b = 0$ are

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2 + 4b}}{2}.$$

Hence,

$$h_k = \begin{cases} \frac{\alpha^k - \beta^k}{\sqrt{a^2+4b}} & \text{if } a^2 + 4b \neq 0, \\ n \cdot \left(\frac{a}{2}\right)^{n-1} & \text{if } a^2 + 4b = 0. \end{cases}$$

Lemma 2.1. *Let a sequence (a_k) be defined by*

$$a_k = \alpha_1 h_{k+1}^2 + \alpha_2 h_{k+1} h_k + \alpha_3 h_k^2 \tag{2.2}$$

for some given integers α_1, α_2, and α_3. Then the ordinary generating function of a_k is

$$\sum_{k \geq 0} a_k x^k = \frac{\alpha_1 + (-b\alpha_1 + a\alpha_2 + \alpha_3)x - b\alpha_3 x^2}{1 - (a^2 + b)x - (b^2 + a^2 b)x^2 + b^3 x^3}. \tag{2.3}$$

The above result is easily derived by the generating functions of h_k^2, h_{k+1}^2, and $h_k h_{k+1}$.

Lemma 2.2. *Let a sequence (a_k) be defined by*

$$a_k = h_{k+1}^2 - ah_{n+1}h_n - bh_n^2. \tag{2.4}$$

Then

$$a_k = (-b)^k. \tag{2.5}$$

Proof. We rewrite the sequence (a_k) as

$$\begin{aligned} a_k &= h_{k+1}^2 - h_k(ah_{k+1} + bh_k) \\ &= h_{k+1}^2 - h_k h_{k+2} \\ &= h_{k+1}(ah_k + bh_{k-1}) - h_k(ah_{k+1} + bh_k) \\ &= (-b)(h_k^2 - h_{k-1}h_{k+1}). \end{aligned}$$

Repeat the above step k times,

$$a_k = (-b)^k (h_1^2 - h_0 h_2) = (-b)^k.$$

Thus we complete the proof. □

3. The Algorithm

The following algorithm says that if we find a special kind of a Diophantine equation then we can get an identity like those of Ramanujan and McLaughlin.

Algorithm:

Step 1: Find a Diophantine equation of the form

$$x_1^n + x_2^n + \cdots + x_\ell^n = y_1^n + y_2^n + \cdots + y_\ell^n, \tag{3.1}$$

which has a two-parametric quadratic solution

$$\begin{cases} x_i &= \alpha_{i1}p^2 + \alpha_{i2}pq + \alpha_{i3}q^2, \\ y_i &= \beta_{i1}p^2 + \beta_{i2}pq + \beta_{i3}q^2. \end{cases} \tag{3.2}$$

In particular, there exists a y_j such that

$$y_j = p^2 - apq - bq^2. \tag{3.3}$$

We arrange this j to be the last index ℓ.

Step 2: Let a sequence (h_k) be defined by

$$h_0 = 0, h_1 = 1, h_{k+2} = ah_{k+1} + bh_k, \quad \text{for all } k \geq 0, \tag{3.4}$$

where a and b are nonzero integers. Let $p = h_{k+1}$, $q = h_k$. Then the solutions x_i, y_i become the sequences

$$\begin{cases} a_{i,k} &= \alpha_{i1}h_{k+1}^2 + \alpha_{i2}h_{k+1}h_k + \alpha_{i3}h_k^2, \\ b_{i,k} &= \beta_{i1}h_{k+1}^2 + \beta_{i2}h_{k+1}h_k + \beta_{i3}h_k^2, \end{cases} \tag{3.5}$$

which satisfies the identity

$$a_{1,k}^n + a_{2,k}^n + \cdots + a_{\ell,k}^n = b_{1,k}^n + b_{2,k}^n + \cdots + b_{\ell,k}^n. \tag{3.6}$$

Step 3: By Lemma 2.2 the sequence

$$b_{\ell,k} = (-b)^k.$$

Therefore, equation (3.6) becomes

$$a_{1,k}^n + a_{2,k}^n + \cdots + a_{\ell,k}^n = b_{1,k}^n + b_{2,k}^n + \cdots + b_{\ell-1,k}^n + (-b)^{kn}. \tag{3.7}$$

Step 4: Using Lemma 2.1 we get the ordinary generating functions of $a_{i,k}$ and $b_{j,k}$, where $1 \leq i \leq \ell$, $1 \leq j \leq \ell - 1$. □

Further examples of identities like those of Ramanujan and McLaughlin could easily be given, but we turn instead to a different type of identity, Ramanujan's $6-10-8$ identity and Hirschhorn's $3-7-5$ identity.

4. Ramanujan's 6-10-8 Identity

Ramanujan's 6-10-8 identity is one of the most remarkable identities. That is

$$64F_6F_{10} = 45F_8^2, \tag{4.1}$$

where

$$F_k = (a+b+c)^k + (b+c+d)^k + (a-d)^k - (a+c+d)^k - (a+b+d)^k - (b-c)^k, \tag{4.2}$$

and $ad = bc$. On the other hand, Hirschhorn's 3-7-5 identity, which was inspired by Ramanujan, is also very fascinating. It is

$$25H_3H_7 = 21H_5^2, \tag{4.3}$$

where

$$H_k = (a+b+c)^k - (b+c+d)^k - (a-d)^k + (a+c+d)^k - (a+b+d)^k + (b-c)^k, \quad (4.4)$$

and $ad = bc$. Here we present two new similar identities.

Example 4.1. *Let the sequences of integers (a_k), (b_k), (c_k), (d_k), and (e_k) be defined by*

$$\sum_{k\geq 0} a_k x^k = \frac{2+84x+2x^2}{1-99x+99x^2-x^3}, \quad \sum_{k\geq 0} b_k x^k = \frac{3-14x+3x^2}{1-99x+99x^2-x^3},$$
$$\sum_{k\geq 0} c_k x^k = \frac{1+102x+x^2}{1-99x+99x^2-x^3}, \quad \sum_{k\geq 0} d_k x^k = \frac{2-76x+2x^2}{1-99x+99x^2-x^3}, \quad (4.5)$$
$$\sum_{k\geq 0} e_k x^k = \frac{3+26x+3x^2}{1-99x+99x^2-x^3}.$$

Then we have

$$64(1+a_k^6+b_k^6-c_k^6-d_k^6-e_k^6)(1+a_k^{10}+b_k^{10}-c_k^{10}-d_k^{10}-e_k^{10}) = 45(1+a_k^8+b_k^8-c_k^8-d_k^8-e_k^8)^2, \quad (4.6)$$

$$25(1+a_k^3-b_k^3-c_k^3-d_k^3+e_k^3)(1+a_k^7-b_k^7-c_k^7-d_k^7+e_k^7) = 21(1+a_k^5-b_k^5-c_k^5-d_k^5+e_k^5)^2. \quad (4.7)$$

Proof. We use the two-parametric quadratic forms for F_n and H_n (ref. [4]):

$$F_n = (p^2-10pq+q^2)^n + (2p^2+8pq+2q^2)^n + (3p^2-2pq+3q^2)^n - (p^2+10pq+q^2)^n - (2p^2-8pq+2q^2)^n - (3p^2+2pq+3q^2)^n,$$
$$H_n = (p^2-10pq+q^2)^n + (2p^2+8pq+2q^2)^n - (3p^2-2pq+3q^2)^n - (p^2+10pq+q^2)^n - (2p^2-8pq+2q^2)^n + (3p^2+2pq+3q^2)^n.$$

The recursive relation of h_k is defined by

$$h_{k+2} = 10h_{k+1} - h_k.$$

We follow the steps in our algorithm and get the results. □

References

[1] M. D. Hirschhorn, *An amazing identity of Ramanujan*, Math. Magazine, **68.3** (1995), 199–201.
[2] M. D. Hirschhorn, *A proof in the spirit of Zeilberger of an amazing identity of Ramanujan*, Math. Magazine, **69.4** (1996), 267–269.
[3] J. McLaughlin, *An identity motivated by an amazing identity of Ramanujan*, The Fibonacci Quarterly, **48.1** (2010), 34–38.
[4] T. Piezas III, *Ramanujan and the quartic equation* $2^4+2^4+3^4+4^4+4^4=5^4$, `http://www.oocities.org/titus_piezas/RamQuad.htm`, 2005.
[5] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers, With an Introduction by George E. Andrews*, Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988, xxviii+419pp.

MSC2010: 11A55

Department of Mathematics, Taipei Municipal University of Education, No. 1, Ai-Guo West Road, Taipei 100, Taiwan, R.O.C.
E-mail address: kwchen@tmue.edu.tw