

# THE ORDER OF APPEARANCE OF POWERS OF FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. Let  $F_n$  be the  $n$ th Fibonacci number. The order of appearance  $z(n)$  of a natural number  $n$  is defined as the smallest natural number  $k$  such that  $n$  divides  $F_k$ . For instance,  $z(F_n) = n$ , for all  $n \geq 3$ . In this paper, among other things, we prove that

$$z(F_n^{k+1}) = \frac{nF_n^k}{2},$$

for all integers  $k \geq 2$  and  $n > 3$  with  $n \not\equiv 3 \pmod{6}$ .

## 1. INTRODUCTION

Let  $(F_n)_{n \geq 0}$  be the Fibonacci sequence given by  $F_{n+2} = F_{n+1} + F_n$ , for  $n \geq 0$ , where  $F_0 = 0$  and  $F_1 = 1$ . These numbers are well-known for possessing amazing properties (consult [5] together with its very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications. We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence  $(L_n)_{n \geq 0}$  which follows the same recursive pattern as the Fibonacci numbers, but with initial values  $L_0 = 2$  and  $L_1 = 1$ .

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let  $n$  be a positive integer number, the *order* (or *rank*) of *appearance* of  $n$  in the Fibonacci sequence, denoted by  $z(n)$ , is defined as the smallest positive integer  $k$ , such that  $n|F_k$  (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about  $z(n)$  in the literature. For instance,  $z(n) < \infty$  for all  $n \geq 1$ . The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [11, p. 300]. Indeed,  $z(m) < m^2 - 1$ , for all  $m > 2$  (see [16, Theorem, p. 52]) and in the case of a prime number  $p$ , one has the better upper bound  $z(p) \leq p + 1$ , which is a consequence of the known congruence  $F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p}$ , for  $p \neq 2, 5$ , where  $\left(\frac{a}{q}\right)$  denotes the Legendre symbol of  $a$  with respect to a prime  $q > 2$ .

In recent papers, the author [6, 7, 8] found explicit formulas for the order of appearance of integers related to Fibonacci numbers, such as  $F_m \pm 1$ ,  $L_n \pm 1$ ,  $F_{mk}/F_k$  and  $\prod_{i=0}^k F_{n+i}$ ,  $k = 1, 2, 3$ . In particular, it was proved that  $z(F_n \pm 1) = \frac{n^2}{2} - 2$ , for  $4 | n$  and  $z(F_n F_{n+1} F_{n+2}) = \frac{n(n+1)(n+2)}{2}$ , if  $2 | n$ .

In this paper, we continue this program and study the order of appearance of powers of Fibonacci and Lucas numbers. Our main results are the following.

**Theorem 1.1.** *We have*

- (i) *If  $n \equiv 3 \pmod{6}$ , then  $z(F_n^2) = nF_n$  and  $z(F_n^{k+1}) = nF_n^k/2$ , for  $k \geq 2$ ;*

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THE FIBONACCI QUARTERLY

(ii) Let  $k$  and  $n > 3$  be integers, with  $k \geq 0$  and  $n \not\equiv 3 \pmod{6}$ , then

$$z(F_n^{k+1}) = nF_n^k.$$

We recall that the  $p$ -adic order,  $\nu_p(n)$ , of  $n$  is the exponent of the highest power of a prime  $p$  which divides  $n$ . To deal with powers of Lucas numbers, a further complication arises when  $\nu_2(n)$  is large, because  $\nu_2(L_n) \in \{0, 1, 2\}$ .

**Theorem 1.2.** *We have*

- (i) If  $k \geq 1$  and  $n \equiv 3 \pmod{6}$ , then  $z(L_n^{k+1}) = nL_n^k$ ;
- (ii) If  $n \equiv 6 \pmod{12}$ , then  $z(L_n^2) = nL_n$ ,  $z(L_n^3) = nL_n^2/2$  and  $z(L_n^{k+1}) = nL_n^k/4$ , for  $k \geq 4$ .
- (iii) If  $n \equiv 0 \pmod{12}$  and  $k \geq \nu_2(n) + 2$ , then

$$z(L_n^{k+1}) = \frac{nL_n^k}{2^{\nu_2(n)+1}}.$$

Note that the preceding item (iii) describes all possibilities for  $n$  divisible by 12. For instance, if  $n \equiv 12 \pmod{24}$  and  $k \geq 4$ , then  $z(L_n^{k+1}) = nL_n^k/8$ . In general, if  $n \equiv 12 \cdot 2^t \pmod{12 \cdot 2^{t+1}}$  and  $k \geq t + 4$ , then  $z(L_n^{k+1}) = nL_n^k/2^{t+3}$ .

We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci and Lucas numbers such as the d’Ocagne’s identity, a multiple-angle formula and a result concerning the  $p$ -adic order of  $F_n$  and  $L_n$ . Along the way, we shall prove some divisibility results among these sequences. The last section will be devoted to the proof of theorems.

2. AUXILIARY RESULTS

Before proceeding further, we recall some facts on Fibonacci numbers for the convenience of the reader.

**Lemma 2.1.** *We have*

- (a)  $F_n | F_m$  if and only if  $n | m$ .
- (b)  $F_{6n+3} \equiv 2 \pmod{4}$ .
- (c)  $\gcd(F_n, F_{n+1}) = 1$ .
- (d)  $F_{2n} = F_n L_n$ .
- (e) If  $d = \gcd(m, n)$ , then

$$\gcd(F_m, L_n) = \begin{cases} L_d, & \text{if } m/d \text{ is even and } n/d \text{ is odd;} \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

(f) (d’Ocagne’s identity)  $(-1)^n F_{m-n} = F_m F_{n+1} - F_n F_{m+1}$ .

(g) (Multiple-angle formula)  $F_{\ell n} = \sum_{i=1}^{\ell} \binom{\ell}{i} F_i F_n^i F_{n-1}^{\ell-i}$ .

(h)  $L_n^2 - 5F_n^2 = 4(-1)^n$ .

(i) (De Polignac’s formula) Let  $n$  and  $p$  be integers, with  $p$  prime. Then

$$\nu_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor,$$

where, as usual,  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ .

Most of the previous items can be proved by using induction together with the well-known Binet’s formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \text{ for } n \geq 0,$$

where  $\alpha = (1 + \sqrt{5})/2$  and  $\beta = (1 - \sqrt{5})/2$ . We remark that Benjamin and Rouse [2, Theorem 3] gave a combinatorial proof for (g) by using the number of ways to tile a board with squares and dominoes. The proof of (i) can be found in [9, Theorem 4.2]. We also refer the reader to [1, 4, 5, 12] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

**Lemma 2.2.** *We have*

- (a) *If  $F_n|m$ , then  $n|z(m)$ .*
- (b) *If  $n|F_m$ , then  $z(n)|m$ .*

*Proof.* For (a), since  $F_n|m|F_{z(m)}$ , by Lemma 2.1 (a), we get  $n|z(m)$ . In order to prove (b), we write  $m = z(n)q + r$ , where  $q$  and  $r$  are integers, with  $0 \leq r < z(n)$ . So, by Lemma 2.1 (f), we obtain

$$(-1)^{z(n)q} F_r = F_m F_{z(n)q+1} - F_{z(n)q} F_{m+1}.$$

Since  $n$  divides both  $F_m$  and  $F_{z(n)q}$ , then it also divides  $F_r$  implying  $r = 0$  (keep in mind the range of  $r$ ). Thus,  $z(n)|m$ . □

**Remark 1.** *Let  $\mathcal{D}_\ell^+$  be the set of all positive divisors of  $\ell$ . As a consequence of Lemma 2.2 (b), one has that if  $p$  is prime, then  $z(n) = p$ , for all  $n \in \mathcal{D}_{F_p}^+$ . For instance,  $z(37) = z(113) = 19$ ,  $z(57) = z(2417) = 31$ , and  $z(73) = z(149) = z(2221) = 37$ .*

The  $p$ -adic order of Fibonacci and Lucas numbers was completely characterized, see [3, 10, 13, 15]. For instance, from the main results of Lengyel [10], we extract the following result.

**Lemma 2.3.** *For  $n \geq 1$ , we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 2, & \text{if } n \equiv 3 \pmod{6}; \\ 1, & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$ , and if  $p$  is prime  $\neq 2$  or  $5$ , then

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)} \end{cases}$$

$$\nu_p(L_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv z(p)/2 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv z(p)/2 \pmod{z(p)}. \end{cases}$$

A proof of this result can be found in [10].

An interesting and not so known property of Fibonacci numbers is that  $F_n^{k+1} | F_n F_n^k$ , for all integers  $n$  and  $k$ . It is important to notice that, to the best of our knowledge, this result was first proved by Benjamin and Rouse [2], using a purely combinatorial approach, and a second proof is due to Seibert and Trojovský [14] by using mathematical induction together with an identity for  $F_{ht}/F_t$ . Here, we shall use the Lengyel's Theorem in order to give another proof for this fact.

**Lemma 2.4.** *For any integers  $n \geq 3$  and  $k \geq 0$ , we have*

$$n|z(F_n^{k+1})|nF_n^k.$$

*Proof.* Since  $F_n|F_n^{k+1}$ , Lemma 2.2 (a) yields  $n|z(F_n^{k+1})$ . Now, we need to prove that  $F_n^{k+1}|F_nF_n^k$  and so  $z(F_n^{k+1})|nF_n^k$  (Lemma 2.2 (b)). For that, of course, it suffices to prove that  $\nu_p(F_n^{k+1}) \leq \nu_p(F_nF_n^k)$ , for all prime number  $p$ . For  $p = 5$ , we have

$$\nu_5(F_nF_n^k) = \nu_5(nF_n^k) = \nu_5(n) + k\nu_5(F_n) = (k + 1)\nu_5(F_n) = \nu_5(F_n^{k+1}).$$

For a prime  $p \neq 2$  or  $5$ , we need only to consider the case  $z(p)|n$ . By Lemma 2.3, we get

$$\nu_p(F_nF_n^k) = \nu_p(nF_n^k) + \nu_p(F_{z(p)}) = \nu_p(n) + \nu_p(F_{z(p)}) + \nu_p(F_n^k) = \nu_p(F_n^{k+1}).$$

Thus, all that remains is to check the case  $p = 2$ . For that, let us distinguish three cases:

*Case 1.* If  $n \equiv 3 \pmod{6}$ , then  $2^k|F_n^k$ . Thus,  $nF_n^k/2 \equiv 0 \pmod{12}$ , if  $k \geq 3$ . Therefore,

$$\nu_2(F_nF_n^k) > \nu_2\left(\frac{nF_n^k}{2}\right) = \nu_2(nF_n^k) + 1 = k\nu_2(F_n) + 1 = k + 1 = \nu_2(F_n^{k+1}), \quad (2.1)$$

where we used that  $2 \nmid n$ . For  $k = 2$ ,  $nF_n^2/2 \equiv 6 \pmod{12}$  and so

$$\nu_2(F_nF_n^2) > \nu_2\left(\frac{nF_n^2}{2}\right) = 3 = \nu_2(F_n^3). \quad (2.2)$$

Since  $n \equiv 3 \pmod{6}$ , then  $F_n \equiv 2 \pmod{4}$  (by Lemma 2.1 (b)). Thus,  $nF_n \equiv 6 \pmod{12}$ . Hence,  $\nu_2(F_nF_n) = 3 > 2 = \nu_2(F_n^2)$  which solves the case  $k = 1$ .

*Case 2.* If  $n \equiv 6 \pmod{12}$ , then  $nF_n^k \equiv 0 \pmod{12}$ , because  $3|n$ . Then

$$\nu_2(F_nF_n^k) = \nu_2(nF_n^k) + 2 = k\nu_2(F_n) + 3 = 3(k + 1) = \nu_2(F_n^{k+1}),$$

where we used that  $\nu_2(n) = 1$ , since  $4 \nmid n$ .

*Case 3.* If  $n \equiv 0 \pmod{12}$ , then  $12|nF_n^k$  which yields

$$\nu_2(F_nF_n^k) = \nu_2(nF_n^k) + 2 = \nu_2(n) + k\nu_2(F_n) + 2 = (k + 1)(\nu_2(n) + 2) = \nu_2(F_n^{k+1}).$$

This finishes the proof. □

We note that (2.1) and (2.2) implies the following corollary.

**Corollary 1.** *If  $n \equiv 3 \pmod{6}$  and  $k \geq 2$ , then  $F_n^{k+1}$  divides  $\frac{nF_n^k}{2}$ . In particular,  $z(F_n^{k+1})|nF_n^k/2$ .*

Next, we use the same approach as above to provide a ‘‘Lucas-version’’ of the previous result.

**Proposition 2.5.** *For any non-negative integers  $n$  and  $k$ , we have that*

$$L_n^{k+1}|F_{2nL_n^k}.$$

*In particular,*

$$2n|z(L_n^{k+1})|2nL_n^k.$$

*Proof.* Since  $L_n|F_{2n}$ , we may consider  $k \geq 1$ . By Lemma 2.1 (h),  $5 \nmid L_n$ , for all  $n$ . Thus, it suffices to prove that  $\nu_p(L_n^{k+1}) \leq \nu_p(F_{2nL_n^k})$ , for all prime number  $p \neq 5$ . If  $p = 2$ , then we may suppose that  $3|n$ , thus,  $2nL_n^k \equiv 0 \pmod{12}$  and so, by Lemma 2.3,

$$\nu_2(F_{2nL_n^k}) = \nu_2(2n) + k\nu_2(L_n) + 2 \geq k\nu_2(L_n) + 3 > \nu_2(L_n)(k + 1) = \nu_2(L_n^{k+1}),$$

where we used that  $\nu_2(L_n) \leq 2$ . When  $p \neq 2$  or  $5$ , we can assume that  $n \equiv z(p)/2 \pmod{z(p)}$  (otherwise  $\nu_p(L_n) = 0$ ). Thus,  $p|L_n|F_{2n}$  and so, by Lemma 2.2 (b),  $z(p)|2n|2nL_n^k$  and then we use Lemma 2.3 to get

$$\nu_p(F_{2nL_n^k}) = \nu_p(2nL_n^k) + \nu_p(F_{z(p)}) = \nu_p(L_n^k) + (\nu_p(n) + \nu_p(F_{z(p)})).$$

Note that, again by Lemma 2.3, the expression in parentheses is  $\nu_p(L_n)$  and hence,

$$\nu_p(F_{2nL_n^k}) = \nu_p(L_n^k) + \nu_p(L_n) = \nu_p(L_n^{k+1}).$$

□

**Corollary 2.** *If  $n \equiv 3 \pmod{6}$  and  $k \geq 1$ , then  $L_n^{k+1}$  divides  $F_{nL_n^k}$ . In particular,  $z(L_n^{k+1})|nL_n^k$ .*

Now we shall present a proposition whose proof is due to Carlos Gustavo Moreira who communicated his nice proof by e-mail.

**Proposition 2.6.** *Let  $\ell$  be a positive integer and set*

$$d := \gcd \left( F_n^k, \binom{\ell}{1} F_{n-1}^{k-1} + \binom{\ell}{2} F_{n-1}^{k-2} F_n + \cdots + \binom{\ell}{k} F_k F_n^{k-1} \right).$$

*Then  $\nu_p(d) \leq \nu_p(\ell)$ , for all odd primes  $p$ . In particular, we have that  $d|\ell$ , if  $\ell$  is odd.*

*Proof.* To simplify, we set

$$d(\ell, k, n) := \sum_{j=1}^k \binom{\ell}{j} F_j F_n^{j-1} F_{n-1}^{k-j}.$$

If  $p|d$ , then  $p$  divides both  $F_n$  and  $d(\ell, k, n)$ . Thus,  $p|\ell$ , since  $\gcd(p, F_{n-1}^{k-1}) = 1$ . Suppose then that  $\nu_p(\ell) = r > 0$ , so one needs to prove that  $r \geq \nu_p(d)$ . First, note that  $\nu_p(\ell F_{n-1}^{k-1}) = r$ , then it suffices to prove that  $p^{r+1}$  divides each term  $\binom{\ell}{j} F_j F_n^{j-1} F_{n-1}^{k-j}$ , for  $j \geq 2$ , because in this case  $\nu_p(d(\ell, k, n)) = r$  and then  $\nu_p(d) \leq r$ , as desired. In fact, since  $p|F_n$ , then  $\nu_p(F_n^{j-1}) \geq j - 1$ . Now, we use Lemma 2.1 (i) to get

$$\nu_p(j!) = \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{j}{p^2} \right\rfloor + \cdots \leq \frac{j}{p-1} \leq \frac{j}{2},$$

where we used that  $p > 2$ . Since  $\nu_p(j!)$  is an integer, one has that  $\nu_p(j!) \leq (j - 1)/2$  and therefore  $\nu_p(j!) \leq \lfloor (j - 1)/2 \rfloor$ . Hence,

$$\nu_p \left( \binom{\ell}{j} F_j F_n^{j-1} F_{n-1}^{k-j} \right) \geq \nu_p(\ell) - \nu_p(j!) + \nu_p(F_n^{j-1}) \geq r + j - 1 - \left\lfloor \frac{j-1}{2} \right\rfloor \geq r + 1,$$

where in the last inequality we used that  $j - 1 \geq \lfloor (j - 1)/2 \rfloor + 1$ , for all  $j \geq 2$ . This finishes the proof. □

As an immediate consequence of the previous proposition, we deduce the following result.

**Corollary 3.** *If  $F_n^{k+1}|F_{\ell n}$  and  $\ell$  is odd, then  $F_n^k|\ell$ .*

*Proof.* If  $F_n^{k+1}|F_{\ell n}$ , we use Lemma 2.1 (g), to write

$$\frac{F_{\ell n}}{F_n^{k+1}} = \sum_{i=1}^{\ell} \binom{\ell}{i} F_i F_n^{i-(k+1)} F_{n-1}^{\ell-i}$$

and conclude that  $\sum_{i=1}^k \binom{\ell}{i} F_i F_n^{i-(k+1)} F_{n-1}^{\ell-i}$  must be integer. On the other hand, we can rewrite this sum as

$$\frac{F_{n-1}^{\ell-k}}{F_n^k} \cdot \left( \binom{\ell}{1} F_{n-1}^{k-1} + \binom{\ell}{2} F_{n-1}^{k-2} F_n + \dots + \binom{\ell}{k} F_k F_n^{k-1} \right) = \frac{F_{n-1}^{\ell-k} \cdot d(\ell, k, n)}{F_n^k}, \quad (2.3)$$

where we used the same notation for  $d(\ell, k, n)$  as in the proof of Proposition 2.6. By Lemma 2.1 (c), we deduce that  $F_n^k$  divides  $d(\ell, k, n)$ . Thus, Proposition 2.6 yields  $F_n^k = \gcd(F_n^k, d(\ell, k, n)) | \ell$  and this completes the proof.  $\square$

Now, we are ready to deal with the proof of theorems.

### 3. THE PROOFS

**3.1. The proof of Theorem 1.1.** We use the same argument as in the proof of Lemma 2.4 (the calculation of  $p$ -adic valuation for  $p \neq 2$  prime) together with Corollary 1, to see that  $z(F_n^{k+1}) = nF_n^k/2^s$ , for some  $s \geq 0$ . So, in order to prove (i) for  $k \geq 2$ , it suffices to show that  $F_n^{k+1}$  does not divide  $F_{nF_n^k/4}$  (since  $F_{nF_n^k/4} | F_{nF_n^k/2^s}$ , for all  $s \geq 2$ ). So, we shall prove that  $\nu_2(F_n^{k+1}) > \nu_2(F_{nF_n^k/4})$ , for all  $k \geq 2$ .

As  $\nu_2(F_n^3) = 3 > \nu_2(nF_n^2/4) = 1$ ,  $F_n^3 \nmid F_{nF_n^2/4}$  and so we may assume  $k \geq 3$ . For  $k = 3$ ,  $nF_n^3/4 \equiv 6 \pmod{12}$  and then  $\nu_2(nF_n^3/4) = 3 < 4 = \nu_2(F_n^4)$ . For  $k > 3$ ,  $nF_n^3/4 \equiv 0 \pmod{12}$  and thus,

$$\nu_2(F_{nF_n^k/4}) = \nu_2(nF_n^k/4) + 2 = \nu_2(F_n^k) - \nu_2(4) + 2 = k < k + 1 = \nu_2(F_n^{k+1}).$$

In the case of  $k = 1$ , we must prove that  $F_n^2 \nmid F_{nF_n/2}$ . In fact, since  $n \equiv 3 \pmod{6}$ , then  $nF_n/2 \equiv 3 \pmod{6}$ . Thus,  $\nu_2(F_{nF_n/2}) = 1 < 2 = \nu_2(F_n^2)$ . Therefore item (i) is proved.

For (ii), we need to prove that  $\nu_2(F_n^{k+1}) > \nu_2(F_{nF_n^k/2})$ , for all  $k \geq 1$ . Since  $n \not\equiv 3 \pmod{6}$ , then  $n = 6t$ , for some integer  $t$ . The proof splits in two cases:

*Case 1.* If  $t$  is odd, then  $n \equiv 6 \pmod{12}$  yielding  $nF_n^k/2 \equiv 0 \pmod{12}$ , for  $k \geq 1$ . Thus, by Lemma 2.3, we obtain

$$\nu_2(F_{nF_n^k/2}) = \nu_2(nF_n^k/2) + 2 = 3k + 2 < 3(k + 1) = \nu_2(F_n^{k+1}),$$

where we used that  $\nu_2(n) = 1$ .

*Case 2.* If  $t$  is even, then  $12 | n$  and hence,

$$\nu_2(F_{nF_n^k/2}) = \nu_2(n) + k(\nu_2(n) + 2) + 1 < (k + 1)(\nu_2(n) + 2) = \nu_2(F_n^{k+1}),$$

and the proof is complete.

**3.2. The Proof of Theorem 1.2.** Analogously to the previous proof, we may deduce that  $z(L_n^{k+1}) = nL_n^k/2^s$ , with  $s \geq 0$ . In order to prove (i), we note that according to Corollary 2, it suffices to prove that  $L_n^{k+1} \nmid F_{nL_n^k/2}$ . Since  $n \equiv 3 \pmod{6}$ , then  $nL_n^k/2 \equiv 0 \pmod{12}$ , for  $k > 1$ . Therefore,

$$\nu_2(F_{nL_n^k/2}) = \nu_2(nL_n^k/2) + 2 = 2k + 1 < \nu_2(L_n^{k+1}).$$

In the case of  $k = 1$ ,  $nL_n/2 \equiv 6 \pmod{12}$  yielding  $\nu_2(F_{nL_n/2}) = 3 < 4 = \nu_2(L_n^2)$ .

Now, we shall prove item (ii) only for the case in which  $k \geq 4$  (the other cases can be handled in a similar way). Since  $n \equiv 6 \pmod{12}$ , then  $12 | nL_n^k/2^s$ , for  $s \in \{2, 3\}$ . Thus,

$$\nu_2(F_{nL_n^k/2^s}) = \nu_2(n) + k - s + 2 = k - s + 3 \leq \nu_2(L_n^{k+1}),$$

where the equality holds if and only if  $s = 2$ . So  $z(L_n^{k+1}) = nL_n^k/4$  as desired.

## ORDER OF APPEARANCE OF POWERS OF FIBONACCI NUMBERS

For the last item, since  $12|n$  then  $nL_n^k/2^s \equiv 0 \pmod{12}$ , for  $s \in \{\nu_2(n) + 1, \nu_2(n) + 2\}$ . Thus, by Lemma 2.3,

$$\nu_2(F_{nL_n^k/2^s}) = \nu_2(n) + k - s + 2.$$

Since  $s \geq \nu_2(n) + 1$ , then

$$\nu_2(F_{nL_n^k/2^s}) = \nu_2(n) + k - s + 2 \leq k + 1 = \nu_2(L_n^{k+1}).$$

Thus  $L_n^{k+1}$  divides  $F_{nL_n^k/2^s}$  if and only if  $s = \nu_2(n) + 1$ . The proof is then complete.  $\square$

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### REFERENCES

- [1] A. Benjamin and J. Quinn, *The Fibonacci numbers—exposed more discretely*, Math. Mag., **76.3** (2003), 182–192.
- [2] A. Benjamin and J. Rouse, *When does  $F_m^L$  divide  $F_n$ ? A combinatorial solution*, Applications of Fibonacci Numbers, **10** (2006), 27–34.
- [3] J. H. Halton, *On the divisibility properties of Fibonacci numbers*, The Fibonacci Quarterly, **4.3** (1966), 217–240.
- [4] D. Kalman and R. Mena, *The Fibonacci numbers—exposed*, Math. Mag., **76.3** (2003), 167–181.
- [5] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.
- [6] D. Marques, *On integer numbers with locally smallest order of appearance in the Fibonacci sequence*, Internat. J. Math. Math. Sci., Article ID 407643 (2011), 4 pages.
- [7] D. Marques, *On the order of appearance of integers at most one away from Fibonacci numbers*, The Fibonacci Quarterly, **50.1** (2012), 36–43.
- [8] D. Marques, *The order of appearance of product of consecutive Fibonacci numbers*, The Fibonacci Quarterly, **50.2** (2012), 132–139.
- [9] I. Niven, H. Zuckerman, and H. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons, Inc., New York, 1991.
- [10] T. Lengyel, *The order of the Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **33.3** (1995), 234–239.
- [11] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math., **1** (1878), 184–240, 289–321.
- [12] P. Ribenboim, *My Numbers, My Friends: Popular Lectures on Number Theory*, Springer-Verlag, New York, 2000.
- [13] D. W. Robinson, *The Fibonacci matrix modulo  $m$* , The Fibonacci Quarterly, **1.2** (1963), 29–36.
- [14] J. Seibert and P. Trojovský, *On divisibility of a relation of the Fibonacci numbers*, Int. J. Pure Appl. Math., **46.3** (2008), 443–448.
- [15] J. Vinson, *The relation of the period modulo  $m$  to the rank of apparition of  $m$  in the Fibonacci sequence*, The Fibonacci Quarterly, **1.2** (1963), 37–45.
- [16] N. N. Vorobiev, *Fibonacci Numbers*, Birkhäuser, Basel, 2003.

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