A REMARK ON THE RADICAL OF ODD PERFECT NUMBERS

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Abstract. If n is an odd perfect number with Euler's prime q, we show that if $3 \nmid n$ and $q \leq 148\,207$ (resp. if $3 \mid n$ and $q \leq 223$), then $\sqrt{n} \geq rad(n)$. We also show the non-existence of odd perfect numbers of certain forms.

1. Introduction

Let n be a positive integer and let $n = \prod p_i^{\alpha_i}$ be its prime factorization. Then $rad(n) := \prod p_i$. The integer n is perfect if $\sigma(n) = 2n$, where σ is the sum of divisors function. In [4] Luca and Pomerance prove that if n is an odd perfect number, then $rad(n) \leq 2n^{17/26}$. By a result of Euler, an odd perfect number (if there exist any) is of the form: $n = q^{4b+1}.\prod p_i^{2a_i}$, with $q \equiv 1 \pmod 4$, the prime q is called the Euler's prime of n. Clearly if $b > 0$, then $\sqrt{n} \geq rad(n)$. Here we show that if $3 \nmid n$ and if q is small ($q \leq 148\,207$), then this inequality holds (Proposition 3.1). We also show a similar result when $3 \mid n$, but with a much weaker bound ($q \leq 223$). Computations are very limited and there is no doubt that with more computational power these results can be improved. By the way, we also prove (Proposition 2.3, Proposition 2.5, Lemma 4.1) the non-existence of odd perfect numbers of certain types.

2. Perfect Numbers of Given Types.

Following Brauer [1], we will use the following result.

Lemma 2.1. *Let p be a positive prime. The Diophantine equation $p^2 + p + 1 = y^m$ has no solution for $m > 1$.*

Proof. See [1]. □

Remark 2.2. (a) *By the way, observe that $(-19)^2 - 19 + 1 = 7^3$.*

(b) *We also have the following well-known fact (p a positive prime): $3^m \mid p^2+p+1 \Leftrightarrow m = 1$ and $p \equiv 1 \pmod 3$. If $q > 3$ is a prime such that $q^m \mid p^2 + p + 1$, then $q \equiv 1 \pmod 3$.*

Proposition 2.3. *Let $n = qr_1^2 r_2^2 \cdots r_l^2 s^{2a}$, $q \equiv 1 \pmod 4$ be the prime factorization of the positive integer n. If $3 \nmid n$ or if $r_1 = 3$, then n is not perfect.*

Proof. Assume n is perfect, then $\sigma(n) = 2n$ and:

$$n = q \cdot s^{2a} \cdot \prod_{i=1}^{l} r_i^2 = \frac{q+1}{2} \cdot \sigma(s^{2a}) \cdot \prod_{i=1}^{l}(r_i^2 + r_i + 1) \tag{1}$$

(1) Assume $3 \nmid n$. In this case $r_i \equiv 2 \pmod 3$, for all i and by (b) of Remark 2.2: $(\prod^l r_i^2, \prod^l (r_i^2 + r_i + 1)) = 1$. It follows from (1) that:

$$\prod_{i=1}^{l}(r_i^2 + r_i + 1) \mid qs^{2a} \tag{2}$$

- If $q \nmid r_i^2 + r_i + 1$, then $r_i^2 + r_i + 1 \mid s^{2a}$ and Lemma 2.1 implies $r_i^2 + r_i + 1 = s$.
- It may happen, but just for one index t, that $q \mid r_t^2 + r_t + 1$. In this case, by the previous step, $r_i^2 + r_i + 1 = s$, if $i \neq t$.

Since $l = \omega(n) - 2$ ($\omega(n)$ the number of prime factors of n) and since $\omega(n) \geq 9$ [5], we get $r_i^2 + r_i + 1 = s = r_j^2 + r_j + 1$ with $i \neq j$, which is impossible. (2) Assume $r_1 = 3$. In this case, for $i > 1$, there are at most two r_i's with $r_i \equiv 1 \pmod 3$. So we may assume that $\prod_{i=4}^{l}(r_i^2 + r_i + 1) \mid qs^{2a}$. Since $l - 3 = \omega(n) - 5 \geq 3$, we conclude as above. □

Remark 2.4. *It is known that no odd perfect number of the form $q^{4b+1}r_1^2r_2^2\cdots r_l^2s^{2a}$ exist if $a = 1$ [6] and $a = 2$ [1, 3].*

In the same vein we have the following proposition.

Proposition 2.5. *Let $n = q \cdot r_1^2 \cdot r_2^2 \cdots r_l^2 \cdot p_1^{2a_1} \cdot p_2^{2a_2}$, $q \equiv 1 \pmod 4$, $1 \leq a_1 \leq a_2$, q, r_i, p_j distinct positive primes. If n is an odd perfect number and if $3 \nmid n$, then $a_1 \geq 3$ and $a_2 \geq 9$.*

Proof. By Proposition 2.3 we know that $a_2 \geq a_1 \geq 2$. We have $r_i \equiv 2 \pmod 3$, for all i, it follows, as in the previous proof, that

$$\prod_{i=1}^{l}(r_i^2 + r_i + 1) \mid qp_1^{2a_1}p_2^{2a_2}. \tag{3}$$

It may happen that for one index i, say $i = 1$, $q \mid (r_1^2 + r_1 + 1)$. In any case we may assume that $\prod_{i=2}^{l}(r_i^2 + r_i + 1) \mid p_1^{2a_1}p_2^{2a_2}$.

If $(r_i^2 + r_i + 1, p_t) = 1$, then by Lemma 2.1, $r_i^2 + r_i + 1 = p_j$, $\{t, j\} = \{1, 2\}$. So we may assume that for $i = 4, \ldots, l$: $r_i^2 + r_i + 1 = p_1^{\alpha_i}p_2^{\beta_i}$, with $\alpha_i \geq 1$, $\beta_i \geq 1$. It follows that $l - 3 \leq \sum_4^l \alpha_i \leq 2a_1$. Since $l - 3 = \omega(n) - 6$ and since $\omega(n) \geq 12$ (see [5]), we get $a_1 \geq 3$. On the other hand, by [2], $\Omega(n) = 2l + 2a_1 + 2a_2 + 1$, the total number of primes dividing n, satisfies $\Omega(n) \geq 75$. It follows that $l - 3 \geq 34 - a_1 - a_2$. Hence, $34 - a_1 - a_2 \leq l - 3 \leq 2a_1$, so $34 \leq a_2 + 3a_1 \leq 4a_2$ and $a_2 \geq 9$. □

3. On the Radical of Odd Perfect Numbers Relatively Prime to 3.

We use the results of the previous section to investigate the radical of odd perfect numbers not divisible by 3. Our result is the following proposition.

Proposition 3.1. *Let $n = q^{4b+1} \cdot \prod p_i^{2a_i}$ be an odd perfect number. Assume $3 \nmid n$ and $q \leq 148\,207$, then $\sqrt{n} \geq rad(n)$.*

Proof. The conclusion is clear if $b > 0$, so let's assume $b = 0$.

Assume there are at least three indices i such that $a_i \geq 2$, say $a_3 \geq a_2 \geq a_1 \geq 2$. Then $n \geq q \cdot p_1^4 \cdot p_2^4 \cdot p_3^4 \cdot \prod_{i>3} p_i^2$, hence, $\sqrt{n} \geq \sqrt{q} \cdot p_1^2 \cdot p_2^2 \cdot p_3^2 \cdot \prod_{i>3} p_i$. We have to show that under our assumptions $p_1^2 \cdot p_2^2 \cdot p_3^2 \geq q$. Since $3 \nmid n$, we have $p_1 \geq 5$, $p_2 \geq 7$, $p_3 \geq 11$ and since $5^2 \cdot 7^2 \cdot 11^2 = 148\,225$, we are done.

If there are less than three indices i such that $a_i \geq 2$, then by Proposition 2.5 $a_1 \geq 3$ and $a_2 \geq 9$ and $n \geq q \cdot p_1^6 \cdot p_2^{18} \cdot \prod_{i>2} p_i^2$ and it is enough to check $p_1^4 \cdot p_2^{16} \geq q$. Since $p_1^4 \cdot p_2^{16} \geq 7^4 \cdot 5^{16} > 148\,207$, we are done. □

4. The Case $3 \mid n$.

If $n = q \cdot 3^{2a} \cdot p_1^2 \cdot p_2^2 \cdots p_l^2$ is an odd perfect number, we know by [1], that $a \geq 3$.

Lemma 4.1. *If $n = q \cdot 3^6 \cdot \prod^t p_i^2$, where q, p_i are distinct primes > 3, then n is not perfect.*

Proof. Assume n is perfect and write it as: $n = q \cdot 3^6 \cdot \prod^k p_j^2 \cdot \prod^l r_i^2$, where $p_j \equiv 1 \pmod 3$, $r_i \equiv 2 \pmod 3$ (and $q \equiv 1 \pmod 4$). From $\sigma(n) = 2n$ we obtain

$$n = q \cdot 3^6 \cdot \prod^k p_j^2 \cdot \prod^l r_i^2 = \frac{q+1}{2} \cdot \prod^k (p_j^2 + p_j + 1) \cdot \prod^l (r_i^2 + r_i + 1) \cdot 1\,093, \tag{4}$$

where $1\,093 = \sigma(3^6)$ is a prime $\equiv 1 \pmod{12}$. Since $p_j \equiv 1 \pmod 3$, $\sigma(p_j^2) = 3.c_j$ where $(3, c_j) = 1$ (see Remark 2.2). It follows that $3^k \mid 3^6$, so $k \leq 6$ and we have:

$$q \cdot 3^{6-k} \cdot \prod^k p_j^2 \cdot \prod^l r_i^2 = \frac{q+1}{2} \cdot \prod^k c_j \cdot \prod^l \sigma(r_i^2) \cdot 1\,093. \tag{5}$$

If $6 - k > 0$, since $(3, c_j) = 1$, $r_i \equiv 2 \pmod 3$ and $1\,093 \equiv 1 \pmod 3$, $3^{6-k} \parallel (q+1)/2$. This implies $q \equiv 2 \pmod 3$. But then $(\sigma(r_i^2), q) = (c_j, q) = 1$ (see Remark 2.2) and $q \neq 1\,093$, so q cannot divide the LHS of (5) contradiction.

This shows $k = 6$, $q \equiv 1 \pmod{12}$, moreover:

$$\prod^l r_i^2 \mid \frac{q+1}{2}. \tag{6}$$

We have $q \neq 1\,093$. Indeed otherwise $(q+1)/2 = 547$ which is a prime $\equiv 1 \pmod 3$, so $p_1 = 547$. Then $\sigma(547^2) = 3 \times 163 \times 613$, so $p_2 = 163$, $p_3 = 613$. Since $\sigma(613^2) = 3 \times 7 \times 17\,923$, $\sigma(163^2) = 3 \times 7 \times 19 \times 67$, we get too many p_j's ($p_4 = 7$, $p_5 = 17\,923$, $p_6 = 19$, $p_7 = 67$).

So we may assume $p_1 = 1\,093$. We have $\sigma(p_1^2) = 3 \times 398\,581$, so $c_1 = 398\,581$ which is a prime $\equiv 1 \pmod{12}$. If $c_1 = q$, then $(q+1)/2 = 17 \times 19 \times 617$. Since $17 \equiv 2 \pmod 3$, $l > 0$ and we get a contradiction with (6). We conclude that $p_2 = 398\,581$. Now $\sigma(p_2^2) = 3 \times 52\,955\,737\,381 = 3 \times 1\,621 \times 32\,668\,561$. Both $1\,621$ and $s_2 := 32\,668\,561$ are primes $\equiv 1 \pmod{12}$.

If $q = 1\,621$, then $p_3 = s_2$ and $(q+1)/2 = 811$ which is prime, so $p_4 = 811$. Now $\sigma(811^2) = 3 \times 31 \times 73 \times 97$ too many p_j's again.

So we may assume $p_3 = 1\,621$. We have $\sigma(p_3^2) = 3 \times 7 \times 13 \times 9\,631$. Since $q \neq 7$, $p_4 = 7$. Then $\sigma(7^2) = 3 \times 19$, $p_5 = 19$ too many p_j's again (one at most among s_2, 13 and $9\,631$ is q). □

To conclude we have the following proposition.

Proposition 4.2. *Let $n = q^{4b+1} \cdot \prod p_i^{2a_i}$ be an odd perfect number. If $q \leq 223$, then $\sqrt{n} \geq rad(n)$.*

Proof. If $3 \nmid n$ use Proposition 3.1. Assume $3 \mid n$ and $b = 0$. Let $n = q \cdot 3^{2a} \cdot \prod_{i=1}^k p_i^{2a_i}$. If $a_2 \geq a_1 \geq 2$, then $n \geq q \cdot 3^2 \cdot p_1^4 \cdot p_2^4 \cdot \prod p_i^2$. We conclude since $p_1^2 \cdot p_2^2 \geq 5^2 \cdot 7^2 > 223 \geq q$.

If $a_1 \geq 2$, $a_i = 1$ for $i > 1$, then $n = q \cdot 3^{2a} \cdot p_1^{2a_1} \cdot \prod p_i^2$. By Proposition 2.3, $a \geq 2$. We conclude since $9 \cdot p_1^2 \geq 9 \cdot 5^2 = 225 > q$.

Finally if $a_i = 1$, for all i, then by Lemma 4.1, $a \geq 8$ and since $3^6 > 223$, we are done. □

These results leave open the following problems: (i) improve these bounds, especially when $3 \mid n$ (feasible with some computational power); (ii) does the inequality $\sqrt{n} \geq rad(n)$ hold for every odd perfect number?

5. Acknowledgement

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References

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