# ON THE DISCREPANCY OF THE VAN DER CORPUT SEQUENCE INDEXED BY FIBONACCI NUMBERS

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Abstract. The van der Corput sequence in base $b$ indexed by the Fibonacci numbers $F_n$ is known to be uniformly distributed modulo one if and only if $b$ is a power of 5. In this paper we show that the discrepancy of this sequence is at most of order $1/\sqrt{N}$.

## 1. Introduction

A sequence $(y_n)$ in the unit-interval $[0,1)$ is said to be *uniformly distributed modulo one* if for all intervals $[a,b) \subseteq [0,1)$ we have

$$\lim_{N\to\infty} \frac{\#\{n\,:\,0 \le n < N, y_n \in [a,b)\}}{N} = b-a. \tag{1.1}$$

A quantitative version of (1.1) can be stated in terms of discrepancy. For a sequence $(y_n)$ in $[0,1)$ the *discrepancy* is defined by

$$D_N(y_n) = \sup_{a\le b} \left| \frac{\#\{n\,:\,0 \le n < N, y_n \in [a,b)\}}{N} - (b-a) \right|,$$

where the supremum is extended over all subintervals $[a,b)$ of $[0,1)$. A sequence is uniformly distributed modulo one if and only if its discrepancy tends to zero as $N$ goes to infinity. Schmidt [10] showed that for any sequence $(y_n)$ in $[0,1)$ we have $ND_N(y_n) \ge \frac{\log N}{66 \log 4}$ for infinitely many values of $N \in \mathbb{N}$. An excellent introduction into this topic can be found in the book of Kuipers and Niederreiter [7] (see also [2]).

A prototype for many uniformly distributed sequences is the van der Corput sequence in base $b$. Throughout the paper let $b \ge 2$ be an integer. The *van der Corput sequence* $(x_n)$ *in base* $b$ is defined by $x_n = \varphi_b(n)$, where for $n \in \mathbb{N}_0$ with base $b$ expansion $n = a_0 + a_1 b + a_2 b^2 + \cdots$ the so-called *radical inverse function* $\varphi_b : \mathbb{N}_0 \to [0,1)$ is defined by

$$\varphi_b(n) = \frac{a_0}{b} + \frac{a_1}{b^2} + \frac{a_2}{b^3} + \cdots.$$

It is well-known that for any base $b \ge 2$ the van der Corput sequence is uniformly distributed modulo one and that $ND_N(x_n) = O(\log N)$, see, for example, [1].

In recent years the distribution properties of subsequences of the van der Corput sequence have been studied, see, for example [6, 5]. In [6, Example 4.8] and in [5, Example 5.1] it has been shown that the subsequence $(x_{F_n})$ of the van der Corput sequence in base $b$ indexed by the Fibonacci numbers $F_n$ is uniformly distributed modulo one if and only if $b$ is a power of 5. Both proofs are based on the fact that the Fibonacci numbers are uniformly distributed modulo $b$ if and only if $b$ is a power of 5 (see [8, 9]).

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## 2. The Result

In this paper we give a quantitative result for the uniform distribution of $(x_{F_n})$ for $b = 5^\ell$ for $\ell \in \mathbb{N}$, in terms of discrepancy.

**Theorem 2.1.** *Let $b = 5^\ell$, let $(x_n)$ be the van der Corput sequence in base $b$ and let $(F_n)$ be the sequence of Fibonacci numbers. Then for any $N \in \mathbb{N}$ we have*

$$D_N(x_{F_n}) < \frac{C_b}{\sqrt{N}},$$

*where $C_b = 2b + \frac{8}{b-1}\sum_{\kappa=1}^{b-1}\frac{1}{\sin(\pi\kappa/b)} = O(b)$.*

For the proof of this result we need some preparation. The following definitions go back to [3, 4, 5]. We refer to these references for more detailed information.

For an integer $b \geq 2$ let $\mathbb{Z}_b = \{z = \sum_{r=0}^{\infty} z_r b^r : z_r \in \{0, \dots, b-1\}\}$ be the set of $b$-adic numbers. $\mathbb{Z}_b$ together with the addition forms an abelian group. The set $\mathbb{N}_0$ of non-negative integers is a subset of $\mathbb{Z}_b$. The *Monna map* $\phi_b : \mathbb{Z}_b \to [0, 1)$ is defined by

$$\phi_b(z) = \sum_{r=0}^{\infty} \frac{z_r}{b^{r+1}} \pmod 1.$$

Note that the radical inverse function $\varphi_b$ is just $\phi_b$ restricted to $\mathbb{N}_0$. We also define the inverse $\phi_b^+ : [0, 1) \to \mathbb{Z}_b$ by

$$\phi_b^+\left(\sum_{r=0}^{\infty}\frac{x_r}{b^{r+1}}\right) = \sum_{r=0}^{\infty} x_r b^r,$$

where we always use the finite $b$-adic representation for $b$-adic rationals in $[0, 1)$.

For $k \in \mathbb{N}_0$ we can define characters $\chi_k : \mathbb{Z}_b \to \{c \in \mathbb{C} : |c| = 1\}$ of $\mathbb{Z}_b$ by

$$\chi_k(z) = \exp(2\pi i \phi_b(k) z),$$

where $i = \sqrt{-1}$. Finally, let $\gamma_k : [0, 1) \to \{c \in \mathbb{C} : |c| = 1\}$ where $\gamma_k(x) = \chi_k(\phi_b^+(x))$.

We have the following general discrepancy bound which is based on the functions $\gamma_k$.

**Lemma 2.2.** *Let $g \in \mathbb{N}$. For any sequence $(y_n)$ in $[0, 1)$ we have*

$$D_N(y_n) \leq \frac{2}{b^g} + \sum_{k=1}^{b^g-1} \rho_b(k)\left|\frac{1}{N}\sum_{n=0}^{N-1}\gamma_k(y_n)\right|,$$

*where $\rho_b(0) = 1$ and $\rho_b(k) = \frac{2}{b^{r+1}\sin(\pi\kappa_r/b)}$ for $k \in \mathbb{N}$ with base $b$ expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$, $\kappa_r \neq 0$.*

For prime numbers $b$ this result is a special case of [3, Theorem 3.6]. Using results from [5] it follows easily that it also holds true for general bases $b \geq 2$.

**Lemma 2.3.** *Let $b = 5^\ell$ and let $k \in \mathbb{N}$ with base $b$ expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$ where $\kappa_r \neq 0$. Let $(x_n)$ denote the van der Corput sequence in base $b$. Then for any $N \in \mathbb{N}$ we have*

$$\left|\sum_{n=0}^{N-1}\gamma_k(x_{F_n})\right| < 4b^{r+1}.$$

*Proof.* Let $\mathrm{e}(x) := \exp(2\pi i x)$. Since $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$ it follows that $\varphi_b(k) = \frac{A_k}{b^{r+1}}$ with $A_k \in \{1, \ldots, b^{r+1} - 1\}$. Hence we have,

$$\sum_{n=0}^{N-1} \gamma_k(x_{F_n}) = \sum_{n=0}^{N-1} \mathrm{e}\left(F_n \phi_b(k)\right) = \sum_{n=0}^{N-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right).$$

The Fibonacci sequence $(F_n)$, considered modulo $b^{r+1}$, has period $4b^{r+1}$ (see [11]) and for each integer $a$ there are exactly 4 solutions of $F_n \equiv a \pmod{b^{r+1}}$ per period (see [9]).

Write $N = 4b^{r+1}M + q$ with $M \in \mathbb{N}_0$ and $q \in \{0, \ldots, 4b^{r+1} - 1\}$. Then we obtain

$$\begin{aligned}
\sum_{n=0}^{N-1} \gamma_k(x_{F_n}) &= \sum_{i=0}^{M-1} \sum_{n=i4b^{r+1}}^{(i+1)4b^{r+1}-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right) + \sum_{n=M4b^{r+1}}^{M4b^{r+1}+q-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right) \\
&= M \sum_{n=0}^{4b^{r+1}-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right) + \sum_{n=0}^{q-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right) \\
&= 4M \sum_{a=0}^{b^{r+1}-1} \mathrm{e}\left(a A_k / b^{r+1}\right) + \sum_{n=0}^{q-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right) \\
&= 0 + \sum_{n=0}^{q-1} \mathrm{e}\left(F_n A_k / b^{r+1}\right)
\end{aligned}$$

and the result follows. □

Now we give the proof of Theorem 2.1.

*Proof.* Using Lemma 2.2 and 2.3 we have

$$\begin{aligned}
D_N(x_{F_n}) &\leq \frac{2}{b^g} + \sum_{k=1}^{b^g-1} \rho_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(x_{F_n}) \right| \\
&< \frac{2}{b^g} + \frac{8}{N} \sum_{r=0}^{g-1} \sum_{k=b^r}^{b^{r+1}-1} \frac{1}{\sin(\pi \kappa_r / b)} \\
&= \frac{2}{b^g} + \frac{8}{N} \sum_{r=0}^{g-1} b^r \sum_{\kappa=1}^{b-1} \frac{1}{\sin(\pi \kappa / b)} \\
&\leq \frac{2}{b^g} + \frac{b^g}{N} \frac{8}{b-1} \sum_{\kappa=1}^{b-1} \frac{1}{\sin(\pi \kappa / b)}.
\end{aligned}$$

The result follows by choosing $g = \lfloor \log_b \sqrt{N} \rfloor$. □

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