HORADAM FUNCTIONS AND POWERS OF IRRATIONALS

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Abstract. This paper generalizes a result of Gerdemann to show (with slight variations in some special cases) that, for any real number m and Horadam function $H_n(A,B,P,Q)$,

$$mH_n(A,B,P,Q) = \sum_{i=h}^{k} t_i H_{n+i}(A,B,P,Q),$$

for two consecutive values of n, if and only if,

$$m = \sum_{i=h}^{k} t_i a^i = \sum_{i=h}^{k} t_i b^i$$

where $a = \frac{P+\sqrt{P^2-4Q}}{2}$ and $b = \frac{P-\sqrt{P^2-4Q}}{2}$. (Horadam functions are defined by: $H_0(A,B,P,Q) = A$, $H_1(A,B,P,Q) = B$, $H_{n+1}(A,B,P,Q) = PH_n(A,B,P,Q) - QH_{n-1}(A,B,P,Q)$.) Further generalizations to the solutions of arbitrary linear recurrence relations are also considered.

1. Introduction and Notation

Horadam functions were first studied by Horadam in [6] and [7]. They can be defined by the following definition.

Definition 1.1. *Let $H_0(A,B,P,Q) = A$, $H_1(A,B,P,Q) = B$, and for $n \geq 1$ let*

$$H_{n+1}(A,B,P,Q) = PH_n(A,B,P,Q) - QH_{n-1}(A,B,P,Q).$$

Special cases include the Lucas functions

$$U_n(P,Q) = H_n(0,1,P,Q) \text{ and } V_n(P,Q) = H_n(2,1,P,Q),$$

the Pell polynomials $P_n(x) = U_n(2x,-1)$, the modified Pell polynomials $q_n(x) = H_n(1,x,2x,-1)$ and $q_n^*(x) = H_n(1,1,2x,-1)$ as well as the Pell numbers $U_n(1,-2)$, the Lucas numbers $V_n(1,-1)$, the Jacobstahl numbers $U_n(1,-2)$ and, of course, the Fibonacci numbers $F_n = U_n(1,-1)$.

Often $H_n(A,B,P,Q)$ will be abbreviated to H_n and $U_n(P,Q)$ to U_n.

There are 84 pages on Horadam functions in OEIS, however most of the functions mentioned specifically have $H_n = U_n$ or U_{n+1}. Two exceptions are $H_n(1,3,-1,1)$, (A048739) and $H_n(1,4,2,-1)$, (A048654).

Usually A, B, P and Q are taken to be integers. Lehmer in [8] does allow P to be the square root of an integer. In most of this paper A, B, P and Q can be arbitrary complex numbers.

Two important functions of P and Q, which appear in the explicit forms of H_n and U_n are now defined.

Definition 1.2. *$a(P,Q)$ and $b(P,Q)$ are the roots of the equation $x^2 - Px + Q = 0$.*

These will usually be written as a and b. If $P^2 - 4Q$ is real and positive these can be written as:

$$a = \frac{P + \sqrt{P^2 - 4Q}}{2}, \qquad b = \frac{P - \sqrt{P^2 - 4Q}}{2}.$$

2. Some Properties of Horadam Functions

We now list some known results. These are as in Horadam [6], except that he only gives some special cases of (ii).

Theorem 2.1.

(i) *If* $n \geq 0$ *and* $P^2 \neq 4Q$, $H_n = \left(\frac{B-Ab}{a-b}\right) a^n + \left(\frac{B-Aa}{b-a}\right) b^n$.
(ii) *If* $n \geq 0$, $H_n(A, B, P, P^2/4) = nB(P/2)^{n-1} - (n-1)A(P/2)^n$.
(iii) *If* $n \geq 0$ *and* $P^2 \neq 4Q$, $U_n = \frac{a^n - b^n}{a-b}$.
(iv) *If* $n \geq 0$, $U_n(P, P^2/4) = n(P/2)^{n-1}$.

We also list some obvious special cases.

Corollary 2.2.

(i) *If* $P^2 \neq 4Q$, $B = Ab$ *and* $n \geq 0, H_n = Ab^n$.
(ii) *If* $P^2 \neq 4Q$, $B = Aa$ *and* $n \geq 0, H_n = Aa^n$.
(iii) *If* $B = (P/2)A$, $P^2 = 4Q$ *and* $n \geq 0, H_n = A(P/2)^n = Aa^n = Ab^n$.
(iv) *If* $P = 0$ *and* $n \geq 0$, $H_{2n} = (-Q)^n A$ *and* $H_{2n+1} = (-Q)^n B$.
(v) *If* $P \neq 0$, $Q = 0$ *and* $n > 0$, $H_n = BP^{n-1}$.
(vi) *If* $P = Q = 0$ *and* $n > 1$, $H_n = 0$.

The next theorem relates Horadam functions to Lucas and other Horadam functions. The $q = 0$ case of (i) also appears in Horadam [6].

Theorem 2.3.

(i) *If* $n > q \geq 0$, $H_n = U_{q+1} H_{n-q} - Q U_q H_{n-q-1}$.
(ii) $H_n(A, B, P, Q) = H_{n-1}(B, BP - AQ, P, Q)$
$\qquad = H_{n-i}(H_i(A, B, P, Q), H_{i+1}(A, B, P, Q), P, Q)$.
(iii) $H_n(A, AP, P, Q) = AU_{n+1}(P, Q)$.
(iv) $kH_n(A, B, P, Q) = H_n(kA, kB, P, Q)$.
(v) $k^n H_n(A, B, P, Q) = H_n(A, kB, kP, k^2 Q)$.

Proof.

(i) By induction on q. If $q = 0$, $H_n = 1 \cdot H_n - Q \cdot 0 \cdot H_{n-1}$. If the result holds for q, then $H_n = U_{q+1}(PH_{n-q-1} - QH_{n-q-2}) - QU_q H_{n-q-1} = U_{q+2} H_{n-q-1} - QU_{q+1} H_{n-q-2}$. So the result holds for all $n > q \geq 0$.
(ii) By the recurrence relation for U_n and (i),

$$\begin{aligned} H_n(A, B, P, Q) &= (BP - AQ)U_{n-1} - BQU_{n-2} \\ &= H_{n-1}(H_1, H_2, P, Q) \\ &= H_{n-2}(H_2, H_3, P, Q) \\ &= \dots \\ &= H_{n-i}(H_i, H_{i+1}, P, Q). \end{aligned}$$

(iii) By (i).
(iv) By Theorem 2.1(i) and (ii).
(v)

$$\begin{aligned} ka(P,Q) &= \frac{kP + \sqrt{(kP)^2 - 4k^2Q}}{2} \\ &= a(kP, k^2Q). \end{aligned}$$

Similarly, $kb(P,Q) = b(kP, k^2Q)$. So if $P^2 \neq 4Q$,

$$\begin{aligned} H_n(A, kB, kP, k^2Q) &= \left(\frac{kB - Akb(P,Q)}{k(a(P,Q) - b(P,Q))}\right) k^n a^n(P,Q) \\ &\quad - \left(\frac{kB - Aka(P,Q)}{k(b(P,Q) - a(P,Q))}\right) k^n b^n(P,Q) \\ &= k^n H_n(A,B,P,Q). \\ H_n(A, kB, kP, k^2P^2/4) &= nBk^n(P/2)^{n-1} - (n-1)Ak^n(P/2)^n \\ &= k^n H_n(A,B,P,P^2/4). \end{aligned}$$

□

The recurrence relation for F_n can be used to define F_n for $n < 0$. We will do the same for H_n when this is possible.

Theorem 2.4. *$H_n(A,B,P,Q)$ can be consistently defined for $n < 0$ using the recurrence relation if and only if*

(i) *$Q \neq 0$, as in Theorem 2.1(i) or (ii), where also, $H_{-n}(A,B,P,Q) = Q^{-n}H_n(A, PA - B, P, Q) = H_n(A, \frac{PA-B}{Q}, P/Q, 1/Q)$.*
(ii) *$Q = 0$, $B = PA$; if $P \neq 0$ by $H_n = P^nA$, if $P = A = 0$ by $H_n = 0$.*

Proof.

(i) If $Q \neq 0$, as $P = a + b$ and $Q = ab$, the recurrence relation gives, if $P^2 \neq 4Q$,

$$H_{n-1} = \frac{H_{n+1} - (a+b)H_n}{-ab} = \left(\frac{B - Ab}{a - b}\right) a^{n-1} + \left(\frac{B - Aa}{b - a}\right) b^{n-1},$$

so, given H_n and H_{n+1}, H_m can be defined for all $m < n$, with the explicit expression of Theorem 2.1(i).

By $Q = ab$ and Theorem 2.1(i),

$$\begin{aligned} Q^{-n}H_n(A, PA - B, P, Q) &= \left(\frac{PA - B - Ab}{a - b}\right) b^{-n} + \left(\frac{PA - B - Aa}{b - a}\right) a^{-n} \\ &= \left(\frac{B - Ab}{a - b}\right) a^{-n} + \left(\frac{B - Aa}{b - a}\right) b^{-n} \\ &= H_{-n}(A,B,P,Q). \end{aligned}$$

If $P^2 = 4Q \neq 0$, $a = b = P/2$ then, by the recurrence relation,

$$\begin{aligned} H_{n-1} &= 2nB(P/2)^{n-2} - 2(n-1)A(P/2)^{n-1} - (n+1)B(P/2)^{n-2} + nA(P/2)^{n-1} \\ &= (n-1)B(P/2)^{n-2} - (n-2)A(P/2)^{n-1}, \end{aligned}$$

so H_m can be defined for $m < n$, with the explicit representation of Theorem 2.1(ii).

$$\begin{aligned} Q^{-n}H_n(A, PA-B, P, P^2/4) &= (P/2)^{-2n}(n(PA-B)(P/2)^{n-1} - (n-1)A(P/2)^n) \\ &= -nB(P/2)^{-n-1} + (n+1)A(P/2)^{-n} \\ &= H_{-n}(A, B, P, Q). \end{aligned}$$

By Theorem 2.3(v), $H_n(A, (PA-B)/Q, P/Q, 1/Q) = Q^{-n}H_n(A, PA-B, P, Q)$.

(ii) If $Q = 0$ and H_n is to be defined for $n < 0$ by the recurrence relation, we must have $B = H_1 = PH_0 - 0H_{-1} = PA$.

If $P \neq 0$, $H_0 = PH_{-1} - 0H_{-2}$ gives $H_{-1} = P^{-1}A$. Similarly, for any $n < 0$, $H_n = P^nA$. If $P = 0$, $H_{-i} = A = PH_{-i-1} - 0H_{-i-2} = 0$, for $i \geq 0$, so $A = B = 0$ and $H_n = 0$ for $n < 0$. □

Horadam [6] also has the $P^2 > 4Q$ and $+\infty$ cases of the following, but gets different results!

Theorem 2.5. *If P and Q are real,*

(i) *$P^2 > 4Q$ and*

(a) *$P > 0$, $\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = a$, and if $Q \neq 0$ and either $A \neq 0$ or $B \neq 0$ is 0, $\lim_{n\to-\infty} \frac{H_{n+1}}{H_n} = b$.*

(b) *$P < 0$, $\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = b$, and if $Q \neq 0$ and either $A \neq 0$ or $B \neq 0$ is 0, $\lim_{n\to-\infty} \frac{H_{n+1}}{H_n} = a$.*

(ii) *If $P^2 = Q > 0$, $\lim_{n\to\pm\infty} \frac{H_{n+1}}{H_n} = P/2 = a = b$.*

(iii) *If $P = 0$, $\frac{H_{2n+1}}{H_{2n}} = \frac{B}{A}$ and $\frac{H_{2n+2}}{H_{2n+1}} = \frac{-QA}{B}$.*

Proof.

(i) If $P^2 > 4Q$,

$$\frac{H_{n+1}}{H_n} = \frac{\left(\frac{B-Ab}{a-b}\right)a^{n+1} + \left(\frac{B-Aa}{b-a}\right)b^{n+1}}{\left(\frac{B-Ab}{a-b}\right)a^n + \left(\frac{B-Aa}{b-a}\right)b^n}.$$

(a) So if $P > 0$, $|a| > |b|$ and $\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = a$, and, provided H_n for $n < 0$ is defined and not identically 0, i.e. $Q \neq 0$ and either $A \neq 0$ or $B \neq 0$ is 0, $\lim_{n\to-\infty} \frac{H_{n+1}}{H_n} = b$.

(b) If $P < 0$, $|b| > |a|$ and $\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = b$, and, provided H_n for $n < 0$ is defined and not identically 0, i.e. $Q \neq 0$ and either $A \neq 0$ or $B \neq 0$ is 0, $\lim_{n\to-\infty} \frac{H_{n+1}}{H_n} = a$.

(ii) If $P^2 = 4Q$,

$$\frac{H_{n+1}}{H_n} = \frac{(n+1)B(P/2)^n - nA(P/2)^{n+1}}{nB(P/2)^{n-1} - (n-1)A(P/2)^n} = \frac{(n+1)BP/2 - nA(P/2)^2}{nB - (n-1)AP/2}.$$

So, $\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = P/2$.

(iii) By Corollary 2.2(iv).

□

Note that Horadam [6] has, for $P^2 > 4Q$:
$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = a$, if $-1 \le b \le 1$, which is equivalent to $-P-1 \le Q$ and either $P < 2$ or $P \ge 2$ and $Q \le P-1$,
and
$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = b$, if $-1 \le a \le 1$, which is equivalent to $P-1 \le Q$ and either $P > -2$ or $Q \le -P-1$ and $P \le -2$.

We can have both conditions holding, for example if $P = 1$ and $Q = 0$, $a = 1$, and $b = 0$, or neither. Another example would be when $P = 3$ and $Q = -5$, $a = \frac{3+\sqrt{29}}{2}$, $b = \frac{3-\sqrt{29}}{2}$.

The following theorem is needed later.

Theorem 2.6.

(i) $H_n - bH_{n-1} = (B - Ab)a^{n-1}$.
(ii) $H_n - aH_{n-1} = (B - Aa)b^{n-1}$.

Proof. (i) If $P^2 \ne 4Q$, $H_n - bH_{n-1} = \left(\frac{B-Ab}{a-b}\right)a^n + \left(\frac{B-Ab}{b-a}\right)a^{n-1}b = (B - Ab)a^{n-1}$.

If $P^2 = 4Q$, $a = b = P/2$ and

$$\begin{aligned} H_n - bH_{n-1} &= nB(P/2)^{n-1} - (n-1)A(P/2)^n - (n-1)B(P/2)^{n-1} + (n-2)A(P/2)^n \\ &= (B - Ab)a^{n-1}. \end{aligned}$$

(ii) Similar.

□

3. Generalizing Gerdemann

Gerdemann's Theorem 1.1 of [3] is a special case of the following theorem.

Theorem 3.1.

(i) *If $P, B \ne 0$ and $B = Aa$,*

$$mH_n = \sum_{i=h}^{k} t_i H_{n+i} \tag{3.1}$$

for any one value of n, if and only if

$$m = \sum_{i=h}^{k} t_i a^i. \tag{3.2}$$

(ii) *If $P, B \ne 0$ and $B = Ab$, equation (3.1) holds for any one value of n, if and only if*

$$m = \sum_{i=h}^{k} t_i b^i. \tag{3.3}$$

(iii) *If $P, B \ne 0$ and $Q = 0$, $a = P$ and $b = 0$ or $b = P$ and $a = 0$ and equation (3.1) holds for any one value of n, if and only if equation (3.2) holds if $a = P$ and equation (3.3) holds if $b = P$.*

(iv) *If $P, Q \ne 0$ and equation (3.1) holds for any two values of n then equations (3.2) and (3.3) hold.*

(v) *If $P, Q \ne 0$, $P^2 - 4Q \ne 0$ or $B = AP/2$, and equations (3.2) and (3.3) hold then equation (3.1) holds.*

Proof.

(i) If $P, B \neq 0$ and $B = Aa$, $a \neq 0$. If $P^2 \neq 4Q$, by Corollary 2.2(ii) and Theorem 2.4(i) and if $P^2 = 4Q$ (as then $a = b$) by Corollary 2.2(iii) and Theorem 2.4(ii), $H_r = Aa^r$ whenever H_r is defined. Clearly equation (3.1) holds if and only if equation (3.2) holds.

(ii) As for (i) with $H_r = Ab^r$.

(iii) If $P \neq 0$ and $Q = 0$, $a = P$ and $b = 0$ or $a = 0$ and $b = P$, so by Corollary 2.2(v) and Theorem 2.4(ii), $H_r = AP^{r-1}$. So equation (3.1) holds if and only if equation (3.2) holds and if and only if $a = P$ and $b = 0$ or if and only if equation (3.3) holds if $b = P$ and $a = 0$.

(iv) Assume that equation (3.1) holds for a particular n and also for some $q < n$. Applying Theorem 2.3(i) to equation (3.1) gives

$$U_{n-q}mH_{q+1} - QU_{n-q+1}mH_q = \sum_{i=h}^{k} j_i(U_{n-q}H_{q+i+1} - QU_{n-q+1}H_{q+i}). \tag{3.4}$$

Adding QU_{n-q+1} times, equation (3.1), with q for n, to this and dividing by U_{n-q} (which is not 0 by Theorems 1(iii) and (iv)), gives equation (3.1) with $q + 1$ for n. Similarly equation (3.1) can be derived whenever all the Horadam functions appearing in it are definable. In particular we have equation (3.1) with $n - 1$ for n. and so, by Theorem 2.6(i), as $a \neq 0$, equation (3.2) holds. Similarly by Theorem 2.6(ii), as $b \neq 0$, equation (3.3) holds.

(v) If $P, Q, P^2 - 4Q \neq 0$, this follows by Theorem 2.1(i) and Theorem 2.4(i). If $P, Q \neq 0, P^2 = 4Q$ and $B = (AP)/2$, it follows by Corollary 2.2(iii) and Theorem 2.4(i).

□

If any of the conditions in the parts of Theorem 3.1 fail, we show that the results will usually fail.

If $B = 0$, H_1 can be added to the right of equation (3.1), but the corresponding a^{1-n} or b^{1-n} cannot be added to the right of equations (3.2) or (3.3). Also with $h = k = 1 - n$ and $t_{1-n} = a$, equation (3.2) is $m = aa^{1-n}$, while $a^{2-n}H_n \neq aH_1$.

If $P = 0$, $a = \sqrt{-Q} = -b$, by Corollary 2.2(iv), equation (3.1) can be $BH_{2n} = AH_{2n+1}$. Equations (3.2) and (3.3) fail as B need not equal $\pm A\sqrt{-Q}$. Also if $-Q = \sqrt{-Q}a$ is equation (3.2) and equation (3.3) is $Q = \sqrt{-Q}b$, then equation (3.1) fails as $-QH_n \neq \pm\sqrt{-Q}H_{n+1}$.

Now we give some more specific examples.

Examples.

1. Let $P = 5$, $Q = 6$, $a = 3$, $b = 2$. So if $B = bA \neq aA$ and $H_n = A2^n$, $14H_n = \sum_{i=1}^{3} H_{n+i}$, while $14 = \sum_{i=1}^{3} 2^i$, but $\sum_{i=1}^{3} 3^i = 39 \neq 14$.
2. Let $P = 1$, $Q = -1$, $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$. So if $B = bA \neq aA$, $H_n = Ab^n$, $2H_n = H_{n+1} + H_{n-2}$, and $2 = a + a^{-2} = b + b^{-2}$.
3. Let $P = Q = 4$, $A = 1$ and $B = 3$, $P^2 = 4Q$, $a = b = 2$. So $B \neq aP/2$. We have $3H_2 = H_3 + H_1 + H_0 = 24$, as equation (3.1) while equations (3.2) and (3.3) fail as $3 \neq 2 + 2^{-1} + 2^{-2}$. Also equations (3.2) and (3.3) can hold as $4 = 2 + 2.2^{-1} + 4.2^{-2}$ while equation (3.1) fails as $4 = 4H_0 \neq H_1 + 2H_{-1} + 4H_{-2} = 31/2$.

4. Let $P = 2i$, $Q = 3\frac{1}{2}$, $a = \left(\frac{2+3\sqrt{2}}{2}\right) i$, $b = \left(\frac{2-3\sqrt{2}}{2}\right) i$. So $7\frac{1}{2}H_n = -H_{n+2} - 7iH_{n-1}$ and $7\frac{1}{2} = -a^2 - 7ia^{-1} = -b^2 - 7ib^{-1}$.

Gerdemann's version of Theorem 3.1 was as follows:

$$mF_n = \sum_{i=h}^{k} F_{n+c_i} <=> m = \sum_{i=h}^{k} \tau^{c_i} \tag{3.5}$$

where $\tau = \frac{1+\sqrt{5}}{2}$.

Gerdemann also showed that, for any integer m, integers h, k, and $c_h, \ldots, c_k$, independent of n, can be found so that the left of this equivalence holds. Hence, any positive integer m can be expressed as a sum of powers of τ.

4. Further Generalization

The anonymous referee suggested that the result could perhaps be generalized to higher order linear recurrences such as:

$$G_n = \sum_{i=n-s}^{n-1} q_{n-i} G_i \tag{4.1}$$

where $G_i = A_i$ for $0 \le i < s$.

Grabner, Tichy, Nemes and Petho [4], in fact, do just that, in the special case where G_n is a Pisot recurrence. This requires $G_0 = 0$ $G_k = q_1 G_{k-1} + \cdots + q_k G_0 + 1$ for $1 \le k < s$ and $q_1 \ge q_2 \cdots \ge q_s$.

Their Lemma 1.1 states that if G_n is a Pisot recurrence,

$$mG_n = \sum_{i=h}^{k} j_i G_{n+i} \tag{4.2}$$

if and only if

$$m = \sum_{i=h}^{k} j_i x^i \tag{4.3}$$

where x is the dominating root of the equation

$$x^s = q_1 x^{s-1} + \cdots + q_{s-1} x + q_s. \tag{4.4}$$

Without G_n being a Pisot recurrence, we can prove the following generalization of Theorem 3.1(iv) and (v).

Theorem 4.1.

(i) *If equation (4.2) is obtained by the recurrence relation (4.1), and x is any root of equation (4.4), then equation (4.3) holds.*

(ii) *If the solutions x of equation (4.4) are all distinct and equation (4.3) holds for all of them, equation (4.2) holds.*

Proof.

(i) By induction on the number p of uses of equation (4.1) in the proof of equation (4.2). If $p = 0, h = k = 0$ and $j_0 = m$, so equation (4.3) holds. If equation (4.2) is obtained by p uses of equation (4.1), equation (4.3) holds, and one further use of the recurrence relation, in the form

$$G_{n+r} = \sum_{i=n+r-s}^{n+r-1} q_{n+r-i} G_i$$

is used, the corresponding version of equation (4.3) is true as the corresponding change requires

$$x^{n+r} = q_1 x^{n+r-1} + \cdots + q_{s-1} x^{n+r+1-s} + q_s x^{n+r-s}.$$

(ii) If equation (4.3) holds for all solutions $x_1, \ldots, x_s$ of equation (4.4) and these solutions are distinct,

$$G_n = k_1 x_1^n + k_2 x_2^n + \cdots + k_s x_s^n \tag{4.5}$$

where $k_1, \ldots, k_s$ are functions of only $G_0, \ldots, G_{s-1}, q_1, \ldots q_s$. Then

$$mG_n = \sum_{i=h}^{k} j_i (k_1 x_1^{i+n} + \cdots + k_s x_s^{i+n})$$

which is equation (4.2).

□

We could also prove counterparts to Theorem 3.1(i), (ii), and (iii) (where not only equation (4.1) is used in the derivation of equation (4.2)), in the case where all the k_i's in equation (4.5), except one, are zero. In view of the examples in Section 3, it is unlikely that much more can be proved, particularly when the roots of equation (4.4) are not all distinct.

There is a lot of literature on expressing integers as sums of (generalized) Horadam functions or powers of rational or irrational numbers, for example Fraenkel [2], Ambroz, Frougny, Masakova, and Pelantova [1] and Hamlin and Webb [5], but only Gerdemann [3] and Grabner, Tichy, Nemes, and Petho [4] have results such as those in Theorems 3.1 and 4.1. The referee provided equation (4.2) for $m = 1$ to 100, for Padovan numbers $P_n = G_n$, as defined above, with $s = 3$, $q_1 = 0, q_2 = q_3 = A_0 = A_1 = A_2 = 1$.

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