A BIJECTION BETWEEN TWO CLASSES OF RESTRICTED COMPOSITIONS

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Abstract. Two proofs, one using generating functions, the other bijective, are given for the following theorem: The number of compositions of n into parts congruent to $1 \pmod{k}$ equals the number of compositions of $n+k-1$ into parts greater than $k-1$. This bijection is then proven to hold for palindromic compositions. A more general theorem is presented in conclusion.

1. Introduction

A *composition* of an integer n is a representation of n as a sum of strictly positive integers called *parts*. For example consider the compositions of 4 listed below:

$$4, \quad 3+1, \quad 1+3, \quad 2+2, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 1+1+1+1$$

Definition 1.1. *A composition, μ, with parts $x_1, x_2, \ldots, x_i$ is represented by $\mu : x_1 + x_2 + \cdots + x_i$.*

Definition 1.2. *Let $C(n, S)$ denote the number of compositions of n with parts taken from the set S.*

Definition 1.3. *Let Z_j be the set of all integers greater than j.*

Definition 1.4. *Let $M_{a,b}$ the set of all positive integers congruent to $a \pmod{b}$.*

Now consider the proof of the following theorem using generating functions.

Theorem 1.5. *Let the integer $k \geq 2$. The number of compositions of n into parts congruent to $1 \pmod{k}$ equals the number of compositions of $n+k-1$ into parts greater than $k-1$.*

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} C(n, Z_{k-1}) x^n &= \sum_{m=1}^{\infty} \left(x^k + x^{k+1} + x^{k+2} + \cdots\right)^m \\
&= \sum_{m=1}^{\infty} \left(\frac{x^k}{1-x}\right)^m \\
&= \frac{x^k}{1-x-x^k} \\
&= x^{k-1} \cdot \frac{x}{1-x-x^k} \\
&= x^{k-1} \cdot \sum_{m=1}^{\infty} \left(\frac{x}{1-x^k}\right)^m \\
&= x^{k-1} \cdot \sum_{m=1}^{\infty} \left(x + x^k + x^{2k} + \cdots\right)^m
\end{aligned}$$

$$\begin{aligned}
&= x^{k-1} \cdot \sum_{m=1}^{\infty} C(n, M_{1,k}) x^m \\
&= \sum_{m=1}^{\infty} C(n, M_{1,k}) x^{m+k-1} \\
&= \sum_{n=k}^{\infty} C(n+k-1, M_{1,k}) x^n.
\end{aligned}$$

□

A bijective proof for Theorem 1.5 is presented in Section 3. We will then examine the bijective map created in the proof of Theorem 1.5 to show that it preserves palindromicity and thereby prove the theorem for palindromic compositions. We will conclude with a more general theorem which encouraged the discovery of the presented results.

2. Definitions

When studying compositions it is useful to write them using a form called binary representation, an idea pioneered by MacMahon (see [2, Sec. IV, Ch. 1, p. 151]). The *binary representation* of a composition is a way of representing the composition in a sequence of zeros and ones. Consider the following composition of seven:

$$2 + 1 + 3 + 1.$$

We can easily represent this composition by the following collection of dots separated by vertical lines:

$$\bullet\bullet\,|\,\bullet\,|\,\bullet\bullet\bullet\,|\,\bullet.$$

If we place a zero between each pair of dots where there is no vertical line and a one in place of each vertical line we have 011001 which is the binary representation of the given composition of seven.

We can now discuss the *conjugate* of a composition, represented by $\bar{\mu}$. Let $B_\mu = b_1 b_2 \ldots b_k$ be the binary representation of a composition μ. The binary representation of the conjugate composition is given by $\bar{B_\mu} = (1-b_1)(1-b_2)\cdots(1-b_k)$. If we consider the composition of seven given in the previous example we find that it has a conjugate composition with a binary representation of 100110 which corresponds to $1+3+1+2$ (see [2, Sec. IV, Ch. 1, p. 151]).

Finally, the *length*, $L(\mu)$, of a composition μ is the number of parts in μ.

3. Finding the Bijection

Lemma 3.1. *Let μ be a composition of an integer n with parts congruent to $1 \pmod{k}$. Let $\bar{\mu} : y_1 + y_2 + \cdots + y_m$ for some positive integer m. Then $y_i = 1$ for every $i \not\equiv 1 \pmod{k}$ and $L(\bar{\mu}) \equiv 1 \pmod{k}$.*

Proof. Let $\mu : x_1 + x_2 + \cdots + x_r$ for some positive integer r. Consider B_μ, the binary representation of μ. As a guiding example consider the following composition of 12 with parts congruent to $1 \pmod{3}$:

$$1 + 1 + 4 + 1 + 1 + 4.$$

Now convert this composition to its appropriate bit map representation:

$$\mu \longrightarrow B_\mu : 1+1+4+1+1+4 \longmapsto 11000111000.$$

Since $x_i \equiv 1 \pmod{k}$ for every i, the zeros in B_μ are in blocks of length congruent to 0 $\pmod{k}$. This is equivalent to saying that the number of zeros in B_μ is a multiple of k. Now take the conjugate of B_μ. For our example this looks like:

$$B_\mu \longrightarrow \bar{B_\mu} : 11000111000 \longmapsto 00111000111.$$

Notice that since the zeros in B_μ were in blocks of length congruent to $0 \pmod{k}$, the ones in $\bar{B_\mu}$ are in blocks of length congruent to $0 \pmod{k}$. We now convert $\bar{B_\mu}$ to its equivalent composition, $\bar{\mu}$.

$$\bar{B_\mu} \longrightarrow \bar{\mu} : 00111000111 \longmapsto 3+1+1+4+1+1+1.$$

Note that blocks of zeros in $\bar{B_\mu}$ can be any possible size but these blocks are mapped to a single integer in $\bar{\mu}$. Since the ones are guaranteed to be in blocks of length congruent to 0 $\pmod{k}$ in $\bar{B_\mu}$, ones will always appear in sequences of length $k-1$ in $\bar{\mu}$. Thus y_i is equal to one for every $i \not\equiv 1 \pmod{k}$.

Finally, note that the number of ones in a binary representation plus one corresponds to the length of a composition. Since the ones in $\bar{B_\mu}$ are in blocks congruent to $0 \pmod{k}$, the total number of ones in $\bar{B_\mu}$ is a multiple of k. This implies that the composition corresponding to $\bar{B_\mu}$ is of length congruent to $1 \pmod{k}$. Since $\bar{B_\mu}$ is the binary representation of $\bar{\mu}$, $L(\bar{\mu}) \equiv 1$ $\pmod{k}$. □

We will now find a bijection between $C(n, Z_{k-1})$ and $C(n+k-1, M_{1,k})$.

Bijective Proof of Theorem 1.5. Let μ be a composition of n with parts congruent to $1 \pmod{k}$. Map μ to its conjugate composition $\bar{\mu}$ through use of the binary representation. Let $\bar{\mu}$: $y_1 + y_2 + \cdots + y_m$. From Lemma 3.1 we have that $L(\bar{\mu}) \equiv 1 \pmod{k}$. Now take the subsequence $\{y_i\}_{i=1}^m$ for $i \equiv 1 \pmod{k}$. We have:

$$\{y_1,\ y_{k+1},\ y_{2k+1},\ \ldots,\ y_{m-k},\ y_m\}.$$

Now add the $k-1$ right adjacent parts to each term in this sequence.

$$\{y_1 + (y_2 + \cdots + y_k),\ \ldots,\ y_{m-k} + (y_{m-(k-1)} + \cdots + y_{m-1}),\ y_m\}.$$

Since $y_i = 1$ for every $i \not\equiv 1 \pmod{k}$ (result of Lemma 3.1), this is the same as adding $k-1$ to every term of the sequence excluding y_m because there are no terms right adjacent to y_m. We now have:

$$\{y_1 + (k-1),\ y_{k+1} + (k-1),\ \ldots,\ y_{m-k} + (k-1),\ y_m\}.$$

We now add $k-1$ to y_m to ensure that y_m is greater than $k-1$:

$$\{y_1 + (k-1),\ y_{k+1} + (k-1),\ \ldots,\ y_{m-k} + (k-1),\ y_m + (k-1)\}.$$

Now each term in this sequence is greater than $k-1$ and the sum of all the terms of the sequence is equal to $n+k-1$ because we only added an additional $k-1$ to the original sum which equaled n. For simplicity rename each term in the sequence $z_1, z_2, \ldots, z_r$. Now $z_1 + z_2 + \cdots + z_r$ is a composition of $n+k-1$ with parts greater than $k-1$.

To map this composition back to the original one simply subtract $k-1$ from each term and the place $k-1$ ones between each part. Then the conjugate of this composition will equal n and will have parts congruent to $1 \pmod{k}$. □

This bijection was inspired by work done in a paper by A. Sills, [3, p. 351-352].

4. Palindromic Compositions

A *palindromic composition* of n is a composition in which the parts are ordered such that they are read the same forward and backwards. The parts of palindromic compositions satisfy the following statement. Let $\mu_p : y_1 + y_2 + \cdots + y_m$ be a palindromic composition. Then $y_{m-j+1} = y_j$ for any $j = 1, 2, \ldots, m$. The palindromic compositions of 4 are:

$$4,\ \ 2+2,\ \ 1+2+1,\ \ 1+1+1+1.$$

These palindromic compositions possess an aesthetically pleasing appearance because of the symmetry that is a direct result of their construction. As several beautiful aspects of nature have been found to be riddled with the Fibonacci sequence one can only hope, or rather expect, that these palindromic compositions may have a Fibonacci-like structure of their own.

Table 1. Palindromic Compositions with Parts Equal to 1 and 2.

n	Palindromic Compositions	Number of Palindromic Compositions
1	1	1
2	$2, 1+1$	2
3	$1+1+1$	1
4	$2+2, 1+2+1, 1+1+1+1$	3
5	$2+1+2, 1+1+1+1+1$	2
6	$2+2+2, 2+1+1+2$ $1+2+2+1, 1+1+2+1+1$ $1+1+1+1+1+1$	5
7	$2+1+1+1+2$ $1+2+1+2+1$ $1+1+1+1+1+1+1$	3
$\vdots$	$\vdots$	$\vdots$

For example, consider Table 1. If we examine the column listing the number of palindromic compositions it seems that it is lining up with two Fibonacci sequences that have been intertwined. One could easily show that this sequence and the interleaved Fibonacci numbers have equivalent generating functions thereby verifying our observation (see Hoggatt [1, p. 352]).

Lemma 4.1. *Let $\{a_i\}$ be a palindromic sequence for $i = 1, 2, \ldots, m$ where $m \equiv 1 \pmod{k}$. Then $\{a_j\}$ for all $j \equiv 1 \pmod{k}$ is a palindromic subsequence.*

Proof. Since $\{a_i\}$ is a palindromic sequence, $a_{m-j+1} = a_j$. Remove all a_i where $i \not\equiv 1 \pmod{k}$. We now have the subsequence:

$$\{a_1,\ a_{k+1},\ a_{2k+1}, \ldots,\ a_{m-k},\ a_m\}.$$

Since $m \equiv 1 \pmod{k}$, there exists some positive integer p such that $m = pk + 1$. We can now rewrite the sequence substituting in $pk + 1$ for m as follows:

$$\{a_1,\ a_{k+1},\ a_{2k+1}, \ldots,\ a_{(p-1)k+1},\ a_{pk+1}\}.$$

Now we need to show that the terms of this subsequence are palindromic. We do this by choosing an arbitrary term in the subsequence, $a_{(p-r)k+1}$, and showing it is equal to a_{rk+1}:

$$\begin{aligned} a_{(p-r)k+1} &= a_{pk-rk+1} \\ &= a_{pk+1-(rk+1)+1} \\ &= a_{m-(rk+1)+1} \\ &= a_{rk+1}. \end{aligned}$$

Thus, each term of the subsequence satisfies the definition of palindromicity. □

Corollary 4.2. *The number of palindromic compositions of* n *into parts congruent to* 1 (mod k) *equals the number of palindromic compositions of* $n+k-1$ *into parts greater than* $k-1$.

Proof. All we need to do is show that the bijective map given in the proof of Theorem 1.5 preserves palindromicity. The first step of the bijection was to take the conjugate composition of μ. By simply examining the definition of the conjugate composition it is clear that $\bar{\mu}$ will be palindromic if μ is palindromic. From Lemma 4.1 we have that the subsequence used in the map is palindromic if μ is palindromic. The final step is equivalent to adding $k-1$ to each term of the subsequence which does not alter palindromicity. Thus the map preserves palindromicity. □

5. Conclusion

Theorem 1.5 is one case of the following previously unpublished theorem found by Sills and myself. Let $C(n;a,b;c)$ denote the number of compositions of n into parts congruent to a (mod b), where each part is greater than or equal to c.

Theorem 5.1. *Suppose* $a \le b$. *The number of compositions of* n *into parts congruent to* a (mod b) *equals the number of compositions of* $n+b-a$ *into parts congruent to* b (mod a) *where each part is greater than* $b-a$.

Proof.

$$\begin{aligned} \sum_{n=a}^{\infty} C(n;a,b;a)x^n &= \sum_{m=1}^{\infty}(x^a + x^{a+b} + x^{a+2b} + \cdots)^m \\ &= \sum_{m=1}^{\infty}(\frac{x^a}{1-x^b})^m \\ &= \frac{\frac{x^a}{1-x^b}}{1-\frac{x^a}{1-x^b}} \\ &= \frac{x^a}{1-x^b-x^a} \\ &= x^{a-b}\cdot\frac{x^b}{1-x^a-x^b} \\ &= x^{a-b}\cdot\sum_{m=1}^{\infty}(\frac{x^b}{1-x^a})^m \end{aligned}$$

$$
\begin{aligned}
&= x^{a-b} \cdot \sum_{m=1}^{\infty} (x^b + x^{b+a} + x^{b+2a} + \cdots)^m \\
&= x^{a-b} \cdot \sum_{n=b}^{\infty} C(n; b, a; b) x^n \\
&= \sum_{n=b}^{\infty} C(n; b, a; b) x^{n+a-b} \\
&= \sum_{n=a}^{\infty} C(n+b-a; b, a; b) x^n.
\end{aligned}
$$

□

Now it is clear that Theorem 1.5 is the case of Theorem 5.1 when $a = 1$. The search remains to find a bijective proof for any such a and b.

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