EXTENDING THE DOMAINS OF DEFINITION OF SOME FIBONACCI IDENTITIES

MARTIN GRIFFITHS

Abstract. In this paper we consider the possibility for extending the domains of definition of particular Fibonacci identities that might initially only be assumed to be valid over the non-negative integers.

1. Introduction

The initial impetus for this article came as we were studying the plethora of results to be found on the authoritative Fibonacci website [4]. Almost 300 formulas involving the golden ratio, the Fibonacci numbers, the Lucas numbers and their generalizations appear there. We noted that one of the results was

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n}, \tag{1.1}$$

but that no corresponding identity involved the related sum

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_k. \tag{1.2}$$

It turns out, as is shown in Section 3, that it is reasonably straightforward to find a formula for (1.2), and indeed this result will almost certainly not be new. The main purpose of this paper, however, is to examine the possibility for extending its domain of definition, and to consider the corresponding situation for other identities.

2. Some Preliminaries

Binet's formula [1, 3, 4] for the kth Fibonacci number is given by

$$F_k = \frac{1}{\sqrt{5}} \left(\phi^k - \left(-\frac{1}{\phi} \right)^k \right), \tag{2.1}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*. Likewise, the kth *Lucas number* is given by

$$L_k = \phi^k + \left(-\frac{1}{\phi} \right)^k. \tag{2.2}$$

Finally, we have the following well-known results:

$$\phi^k = F_k \phi + F_{k-1}, \tag{2.3}$$

and

$$\sqrt{5}\, \phi^k = \phi L_k + L_{k-1}, \tag{2.4}$$

which appear in [4, 5] and [4], respectively. These relations may be proved by indiction or, more directly, by using (2.1) and (2.2).

3. An Initial Result

We now state and prove our first result, noting the slight complication that arises from the necessity for adopting separate formulas for n even and n odd.

Result 3.1. *For any non-negative integer n,*

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k = \begin{cases} 5^{\frac{n}{2}}F_n, & \text{if } n \text{ is even;} \\ 5^{\frac{n-1}{2}}L_n, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, since

$$\sqrt{5}\,\phi = \frac{\sqrt{5}\left(\sqrt{5}+1\right)}{2} = \frac{4+\left(1+\sqrt{5}\right)}{2} = 2+\phi, \tag{3.1}$$

it follows that

$$5^{\frac{n}{2}}\phi^n = (\phi+2)^n = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\phi^k. \tag{3.2}$$

On using (2.3) this leads to

$$\begin{aligned} 5^{\frac{n}{2}}\left(\phi F_n + F_{n-1}\right) &= \sum_{k=0}^{n}\binom{n}{k}2^{n-k}\left(F_k\phi + F_{k-1}\right) \\ &= \phi\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_{k-1}. \end{aligned} \tag{3.3}$$

Since ϕ is irrational, it is the case that $a\phi + b = c\phi + d$ for some $a, b, c, d \in \mathbb{Q}$ if, and only if, $a = c$ and $b = d$. Therefore, when n is even (so that $5^{\frac{n}{2}}$ is rational), (3.3) gives

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k = 5^{\frac{n}{2}}F_n.$$

When n is odd, on the other hand, we may use (2.4) to obtain

$$5^{\frac{n}{2}}\phi^n = 5^{\frac{n-1}{2}}\sqrt{5}\,\phi^n = 5^{\frac{n-1}{2}}\left(\phi L_n + L_{n-1}\right),$$

noting that $5^{\frac{n-1}{2}}$ is rational. From (2.3) and (3.2) it then follows that

$$5^{\frac{n-1}{2}}\left(\phi L_n + L_{n-1}\right) = \phi\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k + \sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_{k-1},$$

and thus, on equating the coefficients of ϕ once more, that

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k = 5^{\frac{n-1}{2}}L_n.$$

□

4. Our Next Result

The method of proof used above means that Result 3.1 is valid for all non-negative integers n. The non-negative integers may therefore be regarded as the domain of the identity in this case. We might now ask ourselves whether or not it is possible in general to extend the domain of such identities from the non-negative integers to the integers. It turns out that direct attempts to do so often result in divergent series. In such cases though, it is sometimes possible to rescue the situation; indeed, at the end of the current section we demonstrate that this is the case for (1.1), for example. For the identity given in Result 3.1, however, it is actually possible to extend the domain in a direct manner, as we now show.

Since $\left|\frac{\phi}{2}\right| < 1$, the following binomial expansion is valid for any $r \in \mathbb{N}$ [6]:

$$\begin{aligned}(2+\phi)^{-r} &= \frac{1}{2^r}\left(1+\frac{\phi}{2}\right)^{-r}\\ &= \frac{1}{2^r}\left(1+\frac{(-r)}{1!}\left(\frac{\phi}{2}\right)+\frac{(-r)(-r-1)}{2!}\left(\frac{\phi}{2}\right)^2+\cdots\right)\\ &= \sum_{k=0}^{\infty}\binom{-r}{k}2^{-r-k}\phi^k, \qquad (4.1)\end{aligned}$$

where

$$\binom{-r}{k} = (-1)^k\binom{r+k-1}{k}$$

by definition [5]. From (3.1) and (4.1) it then follows that

$$\sum_{k=0}^{\infty}\binom{-r}{k}2^{-r-k}\phi^k = 5^{-\frac{r}{2}}\phi^{-r}.$$

Similarly,

$$\left(2-\frac{1}{\phi}\right)^{-r} = \sum_{k=0}^{\infty}\binom{-r}{k}2^{-r-k}\left(-\frac{1}{\phi}\right)^k.$$

Therefore,

$$\sum_{k=0}^{\infty}\binom{-r}{k}2^{-r-k}\left[\phi^k-\left(-\frac{1}{\phi}\right)^k\right] = (2+\phi)^{-r}-\left(2-\frac{1}{\phi}\right)^{-r}.$$

Using Binet's formula (2.1) and the fact that

$$2-\frac{1}{\phi} = \frac{\sqrt{5}}{\phi},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{-r}{k}2^{-r-k}F_k &= \frac{1}{\sqrt{5}}\left[\left(\sqrt{5}\,\phi\right)^{-r}-\left(\frac{\sqrt{5}}{\phi}\right)^{-r}\right]\\ &= -5^{\frac{-r-1}{2}}\left(\phi^r-\frac{1}{\phi^r}\right). \qquad (4.2)\end{aligned}$$

The definitions of the Fibonacci and Lucas sequences may be extended to negative integral subscripts by way of

$$F_{-k} = (-1)^{k+1}F_k \quad \text{and} \quad L_{-k} = (-1)^kL_k,$$

respectively [4]. Therefore, if r is even we may write

$$F_{-r} = -F_r = -\frac{1}{\sqrt{5}}\left(\phi^r - \frac{1}{\phi^r}\right),$$

while if r is odd then

$$L_{-r} = -L_r = -\left(\phi^r - \frac{1}{\phi^r}\right).$$

These results, in conjunction with (2.1), (2.2) and (4.2), lead, on setting $n = -r$, to the following.

Result 4.1. *For any $n \in \mathbb{Z}$,*

$$\sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}F_k = \begin{cases} 5^{\frac{n}{2}}F_n, & \text{if } n \text{ is even;} \\ 5^{\frac{n-1}{2}}L_n, & \text{if } n \text{ is odd.}\end{cases}$$

Note that this is a finite sum when n is a non-negative integer since in this case $\binom{n}{k} = 0$ when $k > n$. Furthermore, Results 3.1 and 4.1 appear identical in every respect other than that the upper limit of the sum in the latter is ∞ (rather than n) in order to cater for the the possibility that n is negative. Let us now return to the comments made in the opening paragraph of this section concerning (1.1), and consider simply replacing the upper limit of n with ∞. Then, if $n = -r$ for some $r \in \mathbb{N}$, we would have

$$\sum_{k=0}^{\infty}\binom{n}{k}2^kF_k = \binom{r-1}{0}F_0 - \binom{r}{1}2F_1 + \binom{r+1}{2}2^2F_2 - \binom{r+2}{3}2^3F_3 + \cdots,$$

which is a divergent series. All is not completely lost, however. By following the procedure adopted in the proof of Result 4.1, this time using the identity $(1+2\phi)^n = \phi F_{3n} + F_{3n-1}$ (which is valid for all $n \in \mathbb{Z}$), we may obtain the result

$$\sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}F_{n-k} = F_{3n} \quad \text{for all } n \in \mathbb{Z}.$$

Therefore, noting that the symmetry of the binomial coefficients implies that

$$\sum_{k=0}^{n}\binom{n}{k}2^kF_k = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_{n-k} = \sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}F_{n-k} \quad \text{for all } n \in \mathbb{N},$$

the right-hand side above may be thought of as a 'continuation' of the left-hand side to the negative integers.

5. Taking Things Further Still

We have thus far extended the domain of our identity from $\mathbb{N}$ to $\mathbb{Z}$. Before going on to consider whether there might be the possibility for extending it to the real numbers $\mathbb{R}$, we briefly demonstrate that Result 4.1 may be generalized in a somewhat different manner. For any $m, n \in \mathbb{Z}$ we have

$$\phi^m(\phi+2)^n = \sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}\phi^{k+m}$$

and

$$\left(-\frac{1}{\phi}\right)^m\left(2-\frac{1}{\phi}\right)^n = \sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}\left(-\frac{1}{\phi}\right)^{k+m},$$

from which we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}\left[\phi^{k+m}-\left(-\frac{1}{\phi}\right)^{k+m}\right] &= \phi^m(\phi+2)^n-\left(-\frac{1}{\phi}\right)^m\left(2-\frac{1}{\phi}\right)^n\\ &= \phi^m\left(\sqrt{5}\,\phi\right)^n-\left(-\frac{1}{\phi}\right)^m\left(\frac{\sqrt{5}}{\phi}\right)^n\\ &= 5^{\frac{n}{2}}\left[\phi^{n+m}-(-1)^n\left(-\frac{1}{\phi}\right)^{n+m}\right].\end{aligned}$$

The result follows.

Result 5.1. *For any $m, n \in \mathbb{Z}$,*

$$\sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}F_{k+m}=\begin{cases}5^{\frac{n}{2}}F_{n+m}, & \text{if } n \text{ is even;}\\ 5^{\frac{n-1}{2}}L_{n+m}, & \text{if } n \text{ is odd.}\end{cases}$$

6. Considering the Reals

Since it is possible to generalize the binomial coefficient [6] by way of

$$\binom{x}{k}=\frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

for any $x \in \mathbb{R}$, noting that

$$\sum_{k=0}^{\infty}\binom{x}{k}q^k=(1+q)^x$$

is valid for all q such that $|q|<1$ [6], we are also able to extend the domain of

$$\sum_{k=0}^{\infty}\binom{n}{k}2^{n-k}F_k$$

to the reals by way of

$$\sum_{k=0}^{\infty}\binom{x}{k}2^{x-k}F_k=2^x\sum_{k=0}^{\infty}\binom{x}{k}\frac{F_k}{2^k}.$$

Now,

$$2^x\sum_{k=0}^{\infty}\binom{x}{k}\left(\frac{\phi}{2}\right)^k=2^x\left(1+\frac{\phi}{2}\right)^x$$

and hence,

$$\sum_{k=0}^{\infty}\binom{x}{k}2^{x-k}\phi^k=5^{\frac{x}{2}}\phi^x. \tag{6.1}$$

Similarly,

$$\sum_{k=0}^{\infty}\binom{x}{k}2^{x-k}\left(-\frac{1}{\phi}\right)^k=\frac{5^{\frac{x}{2}}}{\phi^x}. \tag{6.2}$$

Then (6.1) and (6.2) give

$$\sum_{k=0}^{\infty}\binom{x}{k}2^{x-k}F_k=5^{\frac{x-1}{2}}\left(\phi^x-\frac{1}{\phi^x}\right). \tag{6.3}$$

On setting $h(x) = 5^{\frac{x-1}{2}} \left(\phi^x - \frac{1}{\phi^x}\right)$, and letting $x = n$ for some even $n \in \mathbb{Z}$, the right-hand side of (6.3) is given by

$$h(x) = 5^{\frac{n-1}{2}} \left(\phi^n - \frac{1}{\phi^n}\right) = 5^{\frac{n}{2}} \left[\frac{1}{\sqrt{5}} \left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right)\right] = 5^{\frac{n}{2}} F_n,$$

while, if $x = n$ for some odd $n \in \mathbb{Z}$, it is the case that

$$h(x) = 5^{\frac{n-1}{2}} \left[\phi^n + \left(-\frac{1}{\phi}\right)^n\right] = 5^{\frac{n-1}{2}} L_n.$$

From the above, it follows that the domain of the identity given as Result 4.1 may be extended to the real numbers by way of (6.3).

Taking things a little further, we may extend the domain of the Fibonacci and Lucas numbers to the reals by way of the following generalizations of Binet's formula and the Lucas number formula,

$$F_x = \frac{1}{\sqrt{5}} \left(\phi^x - \frac{\cos \pi x}{\phi^x}\right) \quad \text{and} \quad L_x = \phi^x + \frac{\cos \pi x}{\phi^x}, \tag{6.4}$$

respectively, the former of which appears in [2]. This allows us to extend the two functions on the right-hand side of Result 4.1 to the real numbers via

$$f(x) = 5^{\frac{x}{2}} F_x \quad \text{and} \quad g(x) = 5^{\frac{x-1}{2}} L_x,$$

respectively. It is in fact interesting to observe the relationship between the functions $f(x)$, $g(x)$ and $h(x)$, and this is something we explore briefly.

We know that $f(x)$ and $h(x)$ agree at the even integers. Indeed, this property may be observed in Figure 1, which shows the graphs of these two functions. On inspecting the graphs a little more closely, however, it would seem feasible that not only do the functions take the same values at the even integers but they also appear to possess the same gradients at these particular points. Let us now investigate this possibility.

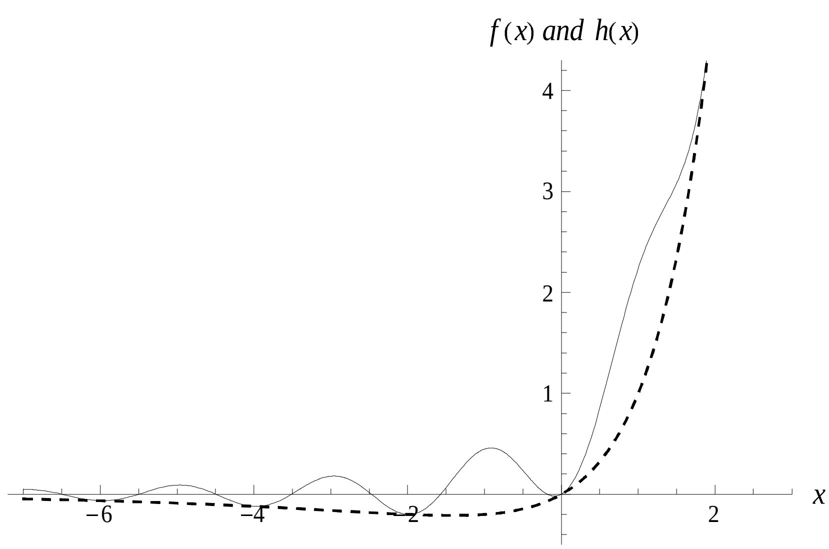

FIGURE 1. The Fibonacci curve (solid) and the curve of $h(x)$ (dashed).

We obtain

$$\frac{dh}{dx} = \frac{5^{\frac{x-1}{2}}}{2\phi^x}\left(\phi^{2x}\log 5\phi^2 + \log\frac{\phi^2}{5}\right)$$

and

$$\frac{df}{dx} = \frac{5^{\frac{x-1}{2}}}{2\phi^x}\left[\sqrt{5}\,\phi^x F_x \log 5 + 2\left(\phi^{2x}\log\phi + \cos\pi x\log\phi + \pi\sin\pi x\right)\right].$$

For the gradients of the curves for $f(x)$ and $h(x)$ to be equal when x is an even integer, $2m$ say, it must be the case that

$$\phi^{4m}\log 5\phi^2 + \log\frac{\phi^2}{5} = \sqrt{5}\,\phi^{2m}F_{2m}\log 5 + 2\phi^{4m}\log\phi + 2\log\phi.$$

By expanding the right-hand side and using (2.1) the above identity is readily established. Then, since $f(2m) = h(2m)$, we see that these curves are actually tangential to each other at the even integers.

A similar calculation may be carried out to show that the gradients of $h(x)$ and $g(x) = 5^{\frac{x-1}{2}}L_x$ are always equal when x is an odd integer, and hence there exists a corresponding tangential property between these graphs at the odd integers. Indeed, note in Figures 1 and 2 the way that the Fibonacci and Lucas curves, respectively, appear to be 'resting' on $h(x)$. In Figure 3 we see how the Fibonacci and Lucas graphs take turns to touch $h(x)$; this 'dove-tailing' and alternating tangential property is displayed very clearly.

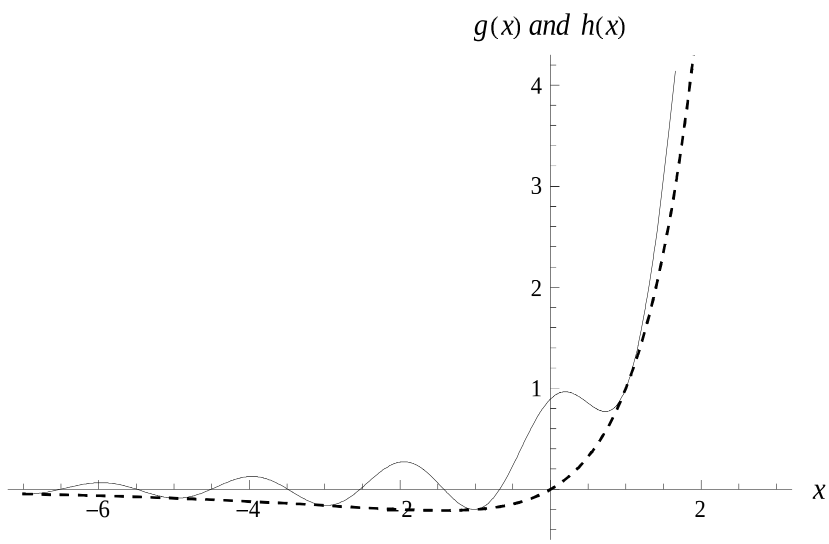

Figure 2. The Lucas curve (solid) and the curve of $h(x)$ (dashed).

7. Closing comments

From the equality $1 + \phi^2 = 2 + \phi$, we may deduce, using the method of Section 3, that

$$\sum_{k=0}^{n}\binom{n}{k}F_{2k} = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_k. \tag{7.1}$$

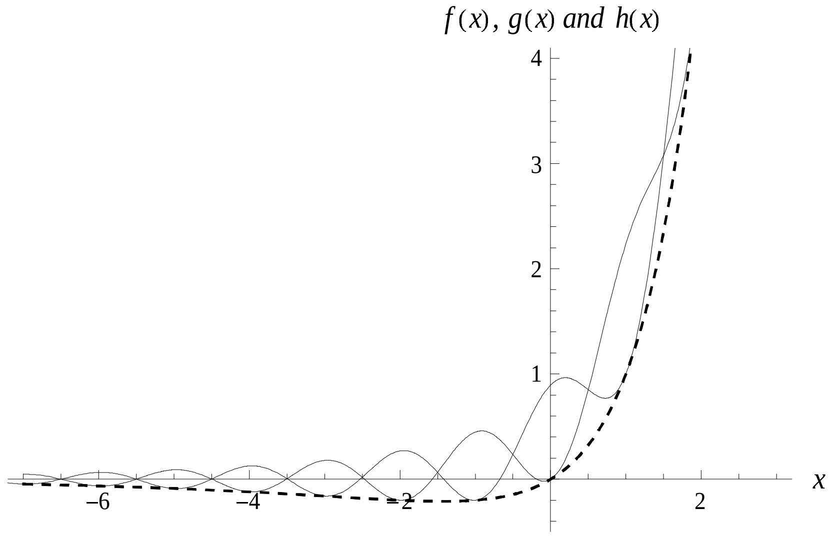

FIGURE 3. The Fibonacci and Lucas curves superimposed on one another, along with the dashed curve $h(x)$.

As was the case with the left-hand side of (1.1), the domain of left-hand side of (7.1) cannot be extended directly to the integers or beyond. However, it can be 'continued' to the reals, either by way of the right-hand side of (7.1) or via $\sum_{k=0}^{n}\binom{n}{k}F_{2(n-k)}$ (see the comments at the end of Section 4). Finally, interested readers might like to show that the domain of the following identity can also be extended to the real numbers:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{2k}F_{2k} = \begin{cases} 5^{\frac{n}{2}}F_{3n}, & \text{if } n \text{ is even;} \\ 5^{\frac{n-1}{2}}L_{3n}, & \text{if } n \text{ is odd.} \end{cases}$$

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MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OX1 3LB, UNITED KINGDOM
E-mail address: martin.griffiths@maths.ox.ac.uk