$\mathcal{C}$-COLOR COMPOSITIONS AND PALINDROMES

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Abstract. An unexpected relationship is demonstrated between n-color compositions (compositions for which a part of size n can take on n colors) and part-products of ordinary compositions. As a consequence, we are able to use techniques developed for studying part-products to generalize the concept of n-color compositions to that of S-restricted $\mathcal{C}$-color compositions, whose part-sizes are restricted to an arbitrary set S of positive integers and for which a part of size n can take on $c_n \in \mathcal{C} = \{c_1, c_2, \ldots\}$ colors. We count the number of S-restricted $\mathcal{C}$-color compositions and the number of $\mathcal{C}$-color palindromic compositions, as well as the total number of parts in each setting. The celebrated Fibonacci numbers persist throughout.

1. Introduction

A composition of ν is a sequence of positive integers, called *parts*, that sum to ν. Recently there has been interest in *n-color compositions*, defined as compositions of ν for which a part of size n can take on n colors [1, 2, 6, 12, 13]. As a brief example, there are eight n-color compositions of 3:

$$(3_1), (3_2), (3_3), (2_1, 1_1), (2_2, 1_1), (1_1, 2_1), (1_1, 2_2), (1_1, 1_1, 1_1).$$

The analogous problem for partitions has been studied to some extent under the alias "n copies of n" [3, 14].

Challenging questions often arise when considering compositions whose part-sizes have been restricted in some way, and there has been recent progress made in this direction for n-color compositions. For example, in [6] the author studies n-color compositions with no parts of size 1, and in [1] the author studies n-color compositions with no parts greater than k. We derive similar results but in a more efficient fashion and in a more general setting. Rather than requiring a part of size n to take on exactly n colors, we allow a part of size n to take on $c_n \in \mathbb{Z}_+$ colors, and rather than forbidding a single part-size or interval of part-sizes, we allow any selection of part-sizes to be forbidden.

We are aided, at least initially, by a rather simple observation.

Theorem 1.1. *The number of n-color compositions of a positive integer ν is equal to the sum of the part-products over all ordinary compositions of ν.*

Proof. For an ordinary composition $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots)$, let $\mathbf{B}(\vec{\lambda}) = \prod_i \lambda_i$. Let Λ_ν be the set of all ordinary compositions of ν and let C_ν denote the set of all n-color compositions of ν. The number of ways to construct an n-color composition having t parts of size $\lambda_1, \lambda_2, \ldots, \lambda_t$ is equal to $\prod_{i=1}^{t} \lambda_i$, since a part of size λ_i can take on λ_i colors. Each choice of part-sizes corresponds to exactly one ordinary composition. Hence,

$$|C_\nu| = \sum_{\vec{\lambda} \in \Lambda_\nu} \prod_{i=1}^{t} \lambda_i = \sum_{\vec{\lambda} \in \Lambda_\nu} \mathbf{B}(\vec{\lambda}) \ . \qquad \square$$

The part-product is studied extensively in [15, 16] where the author shows, for example, that the sum of part-products over all ordinary compositions of ν is $F_{2\nu}$, the 2νth Fibonacci number. It is not difficult to show by other methods that the number of n-color compositions of ν is also $F_{2\nu}$ (see [1] or [8] for a proof); however, the correspondence to part-products provides a new approach to the general study of n-color compositions.

Throughout this paper we will denote the coefficient of x^m in a formal power series $f(x)$ by $[x^m]f(x)$.

2. Part-Size Restrictions and Arbitrary Colorings

In this section, we apply ideas used in [15, 16] in order to extend the concept of n-color compositions. Specifically, the proofs of Theorems 2.1 and 2.3 are generalizations of results in those papers. We define an *S-restricted $\mathcal{C}$-color composition of ν* to be a composition of ν whose parts are in a given set S and for which a part of size n can take on $c_n \in \mathcal{C} = \{c_1, c_2, \ldots, c_\nu\}$ colors, where $c_n \in \mathbb{Z}_+$ for $n = 1, \ldots, \nu$. We denote the set of all such compositions by $C_\nu(S, \mathcal{C})$. If $S = \mathbb{Z}_+$, we write $C_\nu(\mathcal{C})$; if $S = \mathbb{Z}_+$ and $\mathcal{C} = \{1, \ldots, \nu\}$, we write C_ν. We place no restrictions on the choices of $\mathcal{C}$ or S.

Theorem 2.1. *The number of S-restricted $\mathcal{C}$-color compositions of ν is*

$$|C_\nu(S,\mathcal{C})| = [x^\nu]\frac{1}{1 - \sum\limits_{k\in S} c_k x^k} .$$

Proof. For a choice of S and a color-set $\mathcal{C}$, define $G(x) = \sum\limits_{k\in S} c_k x^k$. Then the number of $\mathcal{C}$-color compositions of ν with d parts in S is

$$[x^\nu]G(x)^d .$$

This fact follows because, while the exponents of the multiplied terms add to ν (hence forming compositions of ν), the coefficients multiply to count the number of $\mathcal{C}$-color compositions with those part-sizes. We now sum over all possible values of d to get

$$\sum_{d=1}^{\nu}[x^\nu]G(x)^d = [x^\nu]\frac{1}{1-G(x)} . \qquad \square$$

When $\mathcal{C} = \{1, \ldots, \nu\}$, Theorem 2.1 gives the generating functions

$$\frac{x}{1-3x+x^2} \quad \text{for} \quad S = \mathbb{Z}_+,$$

$$\frac{2x^2-x^3}{1-2x-x^2+x^3} \quad \text{for} \quad S = \mathbb{Z}_+\backslash\{1\},$$

$$\frac{x+x^3}{1-x-2x^2-x^3+x^4} \quad \text{for} \quad S = \mathbb{Z}_{odd,+}.$$

The first of these is derived in [1] while the remaining two are derived in [6]. An asymptotic formula is obtained in [16] for the case when S is an arbitrary cofinite set of positive integers.

Although the number of n-color compositions was originally derived in Theorem 1 of [1], we record it now as an immediate consequence of Theorem 2.1 above.

Corollary 2.2. *The number of n-color compositions of ν is*

$$|C_\nu| = F_{2\nu} .$$

Let $N_\nu(S,\mathcal{C})$ be the total number of parts over all compositions in $C_\nu(S,\mathcal{C})$ and let N_ν be the total number of parts over all n-color compositions of ν.

Theorem 2.3. *The total number of parts over all S-restricted $\mathcal{C}$-color compositions of ν is*

$$N_\nu(S,\mathcal{C}) = [x^\nu]\frac{\sum\limits_{k\in S} c_k x^k}{\left(1-\sum\limits_{k\in S} c_k x^k\right)^2} \ .$$

Proof. Again let $G(x) = \sum\limits_{k\in S} c_k x^k$ and note that, by an argument similar to the proof of Theorem 2.1, the total number of parts over all $\mathcal{C}$-color compositions of ν with parts in S is

$$\sum_{d=1}^{\nu} d\ [x^\nu]G(x)^d = [x^\nu]\frac{G(x)}{(1-G(x))^2} \ . \qquad \square$$

Corollary 2.4. *The number of parts over all n-color compositions of ν is*

$$N_\nu = \tfrac{2\nu}{5}F_{2\nu+1} + \tfrac{2-\nu}{5}F_{2\nu} \ .$$

Proof. In Theorem 2.3, set $S=\mathbb{Z}_+$ and $\mathcal{C}=\{1,\ldots,\nu\}$. Then

$$\begin{aligned} N_\nu &= [x^\nu]\frac{\sum\limits_{k=1}^{\infty} kx^k}{\left(1-\sum\limits_{k=1}^{\infty} kx^k\right)^2} = [x^\nu]\frac{x(1-x)^2}{(1-3x+x^2)^2} \\ &= [x^\nu]\frac{1}{5}\left(\frac{1}{\left(1-\frac{3+\sqrt{5}}{2}x\right)^2} - \frac{1-\frac{2}{\sqrt{5}}}{1-\frac{3+\sqrt{5}}{2}x} + \frac{1}{\left(1-\frac{3-\sqrt{5}}{2}x\right)^2} - \frac{1+\frac{2}{\sqrt{5}}}{1-\frac{3-\sqrt{5}}{2}x}\right) \\ &= \frac{1}{5}\left((\nu+1)\phi^{2\nu} - \left(1-\tfrac{2}{\sqrt{5}}\right)\phi^{2\nu} + (\nu+1)(1-\phi)^{2\nu} - \left(1+\tfrac{2}{\sqrt{5}}\right)(1-\phi)^{2\nu}\right) \\ &= \tfrac{2\nu}{5}F_{2\nu+1} + \tfrac{2-\nu}{5}F_{2\nu} \ . \end{aligned} \qquad \square$$

3. Palindromic Compositions

A *palindromic composition* or *palindrome* [4, 5, 7, 9, 11], also referred to as a *self-inverse composition* [10, 12], is a composition whose part-sequence is the same whether it is read from left to right or right to left. In [7], it is shown that there are $2^{\lfloor \frac{\nu}{2} \rfloor}$ palindromes of ν. The proof relies on a simple argument that fixes the center part λ_i (if there is one) and counts the compositions of $\frac{\nu-\lambda_i}{2}$ that form on either side of the fixed part (or the compositions of $\frac{\nu}{2}$ if there is no center part), as shown in the following example.

Example 3.1.

$\nu = 6$	$\nu = 7$
6	7
1 4 1	**1** 5 1
2 2 2	**2** 3 2
1 1 2 1 1	**1 1** 3 1 1
3 3	**3** 1 3
2 1 1 2	**2 1** 1 1 2
1 2 2 1	**1 2** 1 2 1
1 1 1 1 1 1	**1 1 1** 1 1 1 1

We combine this idea with Theorem 2.1 in order to count the number of $\mathcal{C}$-color palindromes. The analogous results of [12] can then be derived by setting $\mathcal{C} = \{1, \ldots, \nu\}$, as can the analogous results of [7] by setting $\mathcal{C} = \{1, \ldots, 1\}$. Before beginning the proof, we recall several Fibonacci identities that are readily checked by mathematical induction.

Lemma 3.2. *Let F_ν be the νth Fibonacci number ($F_0 = 0$, $F_1 = 1$). The following identities hold for any positive integer m:*

$$(a) \quad \sum_{i=1}^{m} F_{2i} = F_{2m+1} - 1,$$

$$(b) \quad \sum_{i=1}^{m} iF_{2i} = mF_{2m+1} - F_{2m},$$

$$(c) \quad \sum_{i=1}^{m} iF_{2i+1} = mF_{2m+2} - F_{2m+1} + 1,$$

$$(d) \quad \sum_{i=1}^{m} i^2F_{2i} = m^2F_{2m+1} - (2m-1)F_{2m} + 2F_{2m-1} - 2,$$

$$(e) \quad \sum_{i=1}^{m} i^2F_{2i+1} = m^2F_{2m+2} - (2m-1)F_{2m+1} + 2F_{2m} - 1.$$

Let $P_\nu(\mathcal{C})$ be the set of $\mathcal{C}$-color palindromes of ν and let P_ν be the set of n-color palindromes of ν.

Theorem 3.3. *The number of $\mathcal{C}$-color palindromes of ν is*

$$|P_\nu(\mathcal{C})| = \begin{cases} c_\nu + \sum\limits_{k=1}^{\frac{\nu-1}{2}} c_{\nu-2k}|C_k(\mathcal{C})|, & \nu \text{ odd;} \\ \\ c_\nu + \sum\limits_{k=1}^{\frac{\nu-2}{2}} c_{\nu-2k}|C_k(\mathcal{C})| + |C_{\frac{\nu}{2}}(\mathcal{C})|, & \nu \text{ even.} \end{cases}$$

Proof. We directly enumerate the $\mathcal{C}$-color palindromes using the idea from Example 3.1. We first note that the number of $\mathcal{C}$-color palindromes of ν with ν as their center part is c_ν. The number with $(\nu-2)$ as their center part is $c_{\nu-2}$ times the number of $\mathcal{C}$-color compositions of 1. The number with $(\nu-4)$ as their center part is $c_{\nu-4}$ times the number of $\mathcal{C}$-color compositions of 2, and so forth. If ν is even, we have the additional case when there is no center part, in which case we count the number of $\mathcal{C}$-color compositions of $\nu/2$. We combine these cases to get the statement of the theorem. □

In theory, for a given choice of $\mathcal{C}$, these equations can be used either to compute $|P_\nu(\mathcal{C})|$ directly or, in some instances, to derive generating functions for $|P_\nu(\mathcal{C})|$. A different method is used in Theorem 6.2 of [12] to derive the number of n-color palindromes for even values of ν; the number of n-color palindromes for odd values of ν, while not stated directly, is implied in the proof of the same theorem. Nevertheless, we record the result here as an easy consequence of Theorem 3.3 above.

Corollary 3.4. *The number of n-color palindromes of ν is*

$$|P_\nu| = \begin{cases} F_\nu + 2F_{\nu-1}, & \nu \text{ odd;} \\ 3F_\nu, & \nu \text{ even.} \end{cases}$$

We use a similar argument to arrive at the total number of parts over all $\mathcal{C}$-color palindromes. The analogous results of [7] can then be derived by setting $\mathcal{C} = \{1, \ldots, 1\}$. Let $\hat{N}_\nu(\mathcal{C})$ be the number of parts over all $\mathcal{C}$-color palindromes of ν and let $\hat{N}_\nu$ be the number of parts over all n-color palindromes.

Theorem 3.5. *The total number of parts over all $\mathcal{C}$-color palindromes of ν is*

$$\hat{N}_\nu(\mathcal{C}) = \begin{cases} c_\nu + \sum_{k=1}^{\frac{\nu-1}{2}} c_{\nu-2k}(|C_k(\mathcal{C})| + 2N_k(\mathcal{C})), & \nu \text{ odd;} \\ \\ c_\nu + \sum_{k=1}^{\frac{\nu-2}{2}} c_{\nu-2k}(|C_k(\mathcal{C})| + 2N_k(\mathcal{C})) + 2N_{\frac{\nu}{2}}(\mathcal{C}), & \nu \text{ even.} \end{cases}$$

Proof. We again directly enumerate by using the idea from Example 3.1. We first note that the number of parts over all $\mathcal{C}$-color palindromes of ν with ν as their center part is c_ν. The number of parts over all $\mathcal{C}$-color palindromes with $(\nu-2)$ as their center part is $c_{\nu-2}$ times the number of $\mathcal{C}$-color compositions of 1 (counts the center column) plus $c_{\nu-2}$ times twice the number of parts over all $\mathcal{C}$-color compositions of 1 (counts the number of parts on each side), and so forth. If ν is even, we have the additional case when there is no center part, in which case we count the number of parts over all $\mathcal{C}$-color compositions of $\nu/2$ and multiply by two (counts the number of parts on each side). We combine these cases to get the statement of the theorem. □

As with Theorem 3.3, these equations can be used either to compute $\hat{N}_\nu(\mathcal{C})$ directly for a given choice of $\mathcal{C}$ or, in some instances, to derive generating functions for $\hat{N}_\nu(\mathcal{C})$.

Corollary 3.6. *The total number of parts over all n-color palindromes of ν is*

$$\hat{N}_\nu = \begin{cases} \nu F_\nu, & \nu \text{ odd;} \\ \frac{3\nu+2}{5}F_\nu + \frac{6\nu}{5}F_{\nu-1}, & \nu \text{ even.} \end{cases}$$

Proof. Set $\mathcal{C} = \{1, \ldots, \nu\}$ and let ν be odd. Then by an application of Corollaries 2.2 and 2.4 to Theorem 3.5, followed by an application of parts (a) through (e) of Lemma 3.2, we have

$$\begin{aligned}
\hat{N}_\nu &= \nu + \sum_{k=1}^{\frac{\nu-1}{2}} (\nu - 2k)(F_{2k} + 2(\tfrac{2k}{5}F_{2k+1} + \tfrac{2-k}{5}F_{2k})) \\
&= \nu + \sum_{k=1}^{\frac{\nu-1}{2}} (\tfrac{9\nu}{5}F_{2k} - \tfrac{2\nu+18}{5}kF_{2k} + \tfrac{4\nu}{5}kF_{2k+1} + \tfrac{4}{5}k^2F_{2k} - \tfrac{8}{5}k^2F_{2k+1}) \\
&= F_{\nu+1}(\tfrac{2\nu}{5} - \tfrac{2}{5}) + F_\nu(\tfrac{3\nu}{5} - \tfrac{6}{5}) + F_{\nu-1}(2 - \tfrac{2\nu}{5}) + F_{\nu-2}(\tfrac{8}{5}) \\
&= \nu F_\nu.
\end{aligned}$$

Next let ν be even. Similarly,

$$\begin{aligned}
\hat{N}_\nu &= \nu + \sum_{k=1}^{\frac{\nu-2}{2}} (\nu - 2k)(F_{2k} + 2(\tfrac{2k}{5}F_{2k+1} + \tfrac{2-k}{5}F_{2k})) + 2(\tfrac{\nu}{5}F_{\nu+1} + \tfrac{2-\frac{\nu}{2}}{5}F_\nu) \\
&= \nu + \sum_{k=1}^{\frac{\nu-2}{2}} (\tfrac{9\nu}{5}F_{2k} - \tfrac{2\nu+18}{5}kF_{2k} + \tfrac{4\nu}{5}kF_{2k+1} + \tfrac{4}{5}k^2F_{2k} - \tfrac{8}{5}k^2F_{2k+1}) \\
&\qquad + \tfrac{2\nu}{5}F_{\nu+1} + \tfrac{4-\nu}{5}F_\nu \\
&= F_{\nu+1}(\tfrac{2\nu}{5}) + F_\nu(\tfrac{3\nu}{5} - \tfrac{4}{5}) + F_{\nu-1}(\tfrac{2\nu}{5} - \tfrac{2}{5}) + F_{\nu-2}(\tfrac{14}{5} - \tfrac{2\nu}{5}) + F_{\nu-3}(\tfrac{8}{5}) \\
&= \tfrac{3\nu+2}{5}F_\nu + \tfrac{6\nu}{5}F_{\nu-1}.
\end{aligned}$$

□

4. Remarks

While the results of this paper reveal new connections between Fibonacci numbers and integer compositions, they may also bring clarification to some relationships that are known to exist. For example, the binomial identities presented in both [6] and [12], when combined with the results of this paper, are likely to have further combinatorial interpretations and may give rise to some previously undocumented Fibonacci identities.

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