FINITE SUMS IN PASCAL'S TRIANGLE

A. SOFO

Abstract. We consider sums across the nth row in Pascal's triangle and develop their integral identities. In particular we obtain integral identities for $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^q}{(ak+b)^p}$ when $q=-1,0,1,2$.

1. Introduction

It is known that, for any positive integer $n \geq 1$, if we sum across the nth row in Pascal's triangle in the following way

$$S_1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2k+1),$$

then $S_1 = 0$. In fact, for $p \in \mathbb{N}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $a, b \in \mathbb{R}^+$, we have for $n > p$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (ak+b)^p = 0.$$

Dence [7] asked the question and showed that for the reciprocals of the numbers $(2k+1)$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(2k+1)} = \int_0^1 \frac{(1-x)^n}{2\sqrt{x}}\, dx = \frac{4^n (n!)^2}{(2n+1)!}. \tag{1.1}$$

In this paper we will extend the results of Dence and investigate the more general sums

$$S = \sum_{k=0}^{n} (-t)^k \binom{n}{k} \frac{k^q}{(ak+b)^p}, \text{ for } q = -1, 0, 1, 2,$$

$t \in \mathbb{R}^+ \setminus \{0\}$, $p \in \mathbb{N}$ and real positive numbers a and b.

There has recently been renewed interest in the study of series involving binomial coefficients and a number of authors have obtained either closed form representation or integral representation for some particular cases of these series. The interested reader is referred to [1, 2, 3, 4, 5, 10, 15, 17, 18, 21]. The following information and notation will be useful throughout this paper. The generalized hypergeometric representation ${}_pF_q\ [\cdot,\cdot]$, is defined as

$${}_pF_q \left[\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \middle| t \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ (a_2)_n \cdots (a_p)_n\ t^n}{(b_1)_n\ (b_2)_n \cdots (b_q)_n\ n!},$$

where for $w \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $(w)_n$ is Pochhammer's symbol defined by

$$(w)_n = \begin{cases} \frac{\Gamma(w+n)}{\Gamma(n)} = w(w+1)\cdots(w+n-1), \text{ for } n \in \mathbb{N} \\ 1, \text{ for } n = 0, \end{cases}$$

where $\mathbb{Z}_0^-$ denotes the set of non positive integers, $\Gamma(z)$ is the Gamma function, and $B(s,z) = \frac{\Gamma(s)\Gamma(z)}{\Gamma(s+z)}$ is the Beta function for $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(z) > 0$ [13]. The numbers p and q are zero or positive integers (interpreting an empty product as 1) and we assume, for simplicity, that the variable t, the numerator parameters $a_1, a_2, \ldots, a_p$ and the denominator parameters $b_1, b_2, \ldots, b_q$ take on complex values $\mathbb{C}$ provided that no zeros appear in the denominator of ${}_pF_q\ [\cdot,\cdot]$, that is $b_j \notin \mathbb{Z}_0^-$; $j = 1, 2, 3, \ldots, q$. Hence, if a numerator parameter is zero or a negative integer then the hypergeometric series ${}_pF_q\ [\cdot,\cdot]$ terminates, since (see [20])

$$(-n)_j = \begin{cases} 0, \ j > n \\ \frac{(-1)^j \ n!}{(n-j)!}, \ 0 \le j \le n; \ n \in \mathbb{N}. \end{cases}$$

The generalized harmonic numbers of order α are given by

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha} \text{ for } \alpha, n \in \mathbb{N}$$

and for $\alpha = 1$

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \ln(n) \right) = -\psi(1) \approx 0.5772156649\ldots$$

and where $\psi(z)$ denotes the Psi, or digamma function, defined by (see [11])

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right) - \gamma.$$

The polygamma functions are defined by (see [11])

$$\psi^{(k)}(z+1) = \frac{(-1)^k k!}{z^{k+1}} + \psi^{(k)}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1} k!}{(n+z)^{k+1}} = \int_0^\infty \frac{(-1)^{k+1} \ t^k \ e^{-(z+1)t}}{1-e^{-t}} dt$$
$$= \frac{d^{k+1}}{dz^{k+1}} [\ln \Gamma(z+1)] = \frac{d^k}{dz^k} [\psi(z+1)], \ z \ne \{-1, -2, -3, \ldots\}.$$

In the case of non-integer values of the argument $z = \frac{r}{a}$, we may write the generalized harmonic numbers in terms of polygamma functions

$$H_{\frac{r}{a}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!} \psi^{(\alpha)} \left(\frac{r}{a} + 1 \right), \ \frac{r}{a} \ne \{-1, -2, -3, \ldots\}, \tag{1.2}$$

where $\zeta(z)$ is the zeta function. When we encounter harmonic numbers at possible rational values of the argument of the form $H_{\frac{r}{a}}^{(\alpha)}$, they may be evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z = \frac{r}{a}$ (1.2). We also define

$$H_{\frac{r}{a}}^{(1)} = \gamma + \psi\left(\frac{r}{a} + 1 \right) \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [9], (see also [8]), or Choi and Cvijovic [6] in terms of the Polylogarithmic or other special functions. Some specific values are given as

$$H^{(4)}_{\frac{1}{2}} = 16 - 4\zeta(2), \quad H^{(2)}_{\frac{3}{4}} = \frac{16}{9} + 8G - 5\zeta(2),$$
$$H^{(1)}_{-\frac{1}{2}} = -\ln 4, \text{ and } H^{(1)}_{\frac{3}{2}} = \frac{8}{3} - \ln(4).$$

Many others are listed in the book [20].

The following lemma and theorems are the main results presented in this paper.

2. The Main Results

The following lemma will be useful in the proof of the three main theorems.

Lemma 1. *If p, k are positive integers and a and b positive real numbers, then*

$$\begin{aligned} \frac{1}{(ak+b)^p} &= \frac{1}{\Gamma(p)} \int_0^\infty y^{p-1}\ e^{-y(ak+b)} dy \\ &= \frac{(-1)^{p-1}}{a^p\ \Gamma(p)} \int_0^1 x^{\frac{b}{a}-1}\ x^k\ \ln^{p-1}(x)\, dx. \end{aligned} \tag{2.1}$$

Proof. The proof follows upon integration by parts and then by the substitution $x = e^{-ya}$. □

Theorem 1. *Let $t \in \mathbb{R}^+ \backslash \{0\}$, p be a positive integer and for real positive numbers a and b, then*

$$S = \sum_{k=0}^{n} (-t)^k \binom{n}{k} \frac{1}{(ak+b)^p} \tag{2.2}$$

$$= \frac{(-1)^{p-1}}{a^p\Gamma(p)} \int_0^1 (x)^{\frac{b}{a}-1} (1-tx)^n \ln^{p-1}(x)\, dx \tag{2.3}$$

$$= \frac{1}{b^p}\ {}_{p+1}F_p \left[\left. \begin{matrix} \overbrace{\frac{b}{a}, \ldots, \frac{b}{a}}^{p-terms}, -n \\ \underbrace{1+\frac{b}{a}, \ldots, 1+\frac{b}{a}}_{p-terms} \end{matrix} \right| t \right], \tag{2.4}$$

where ${}_pF_q\ [\cdot,\cdot]$ is the generalized hypergeometric function.

Proof. From $S = \sum_{k=0}^{n} (-t)^k \binom{n}{k} \frac{1}{(ak+b)^p}$ and using Lemma 1, we may write

$$S = \sum_{k=0}^{n} (-t)^k \binom{n}{k} \frac{(-1)^{p-1}}{a^p\Gamma(p)} \int_0^1 x^{\frac{b}{a}-1} x^k \ln^{p-1}(x)\ dx.$$

Interchanging the sum and integral produces

$$\begin{aligned} S &= \frac{(-1)^{p-1}}{a^p\Gamma(p)} \int_0^1 x^{\frac{b}{a}-1} \ln^{p-1}(x) \sum_{k=0}^{n} (-t)^k \binom{n}{k} x^k\, dx \\ &= \frac{(-1)^{p-1}}{a^p\ \Gamma(p)} \int_0^1 x^{\frac{b}{a}-1} \ln^{p-1}(x) \sum_{k=0}^{n} (-tx)^k \binom{n}{k}\, dx \end{aligned}$$

and hence,

$$S = \frac{(-1)^{p-1}}{a^p \Gamma(p)} \int_0^1 (x)^{\frac{b}{a}-1} (1-tx)^n \ln^{p-1}(x)\, dx.$$

The identity (2.4) follows upon considering the ratio $\frac{T_{k+1}}{T_k}$ where

$$T_k = (-t)^k \binom{n}{k} \frac{1}{(ak+b)^p}.$$

□

Example 1. *The famous Gauss summation formula is*

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \; for Re(c-a-b) > 0, \; c \notin \mathbb{Z}_0^-$$

and from (2.2) with $p = t = 1$ we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{ak+b} = {}_2F_1 \left[\begin{matrix} \frac{b}{a}, -n \\ 1+\frac{b}{a} \end{matrix} \middle| 1 \right] = \frac{1}{a} B\left(n+1, \frac{b}{a}\right) = \frac{1}{a} \frac{n!}{\left(\frac{b}{a}\right)_{n+1}}.$$

When $a = 2$, $b = 1$ we recover (1.1).

Example 2. *Other interesting cases are $t = 1$ and $p = 2, 3,$ and 4.*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(ak+b)^2} &= \frac{-1}{a^2} \int_0^1 x^{\left(\frac{b}{a}-1\right)} (1-x)^n \ln(x)\, dx \\ &= \frac{1}{b^2} \; {}_3F_2 \left[\begin{matrix} \frac{b}{a}, \frac{b}{a}, -n \\ 1+\frac{b}{a}, 1+\frac{b}{a} \end{matrix} \middle| 1 \right] \\ &= \frac{n!}{a^2 \left(\frac{b}{a}\right)_{n+1}} \left(H^{(1)}_{n+\frac{b}{a}} - H^{(1)}_{\frac{b}{a}-1} \right). \end{aligned} \tag{2.5}$$

The infinite case yields the result

$$\sum_{k=0}^{\infty} \binom{n}{k} \frac{(-1)^k}{(ak+b)^2} = \frac{1}{a^2} B\left(\frac{b}{a}, n+1\right) \left(H^{(1)}_{n+\frac{b}{a}} - H^{(1)}_{\frac{b}{a}-1} \right),$$

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(ak+b)^3} &= \frac{1}{2a^3} \int_0^1 x^{\left(\frac{b}{a}-1\right)} (1-x)^n \ln^2(x)\, dx \\ &= \frac{1}{b^3} \; {}_4F_3 \left[\begin{matrix} \frac{b}{a}, \frac{b}{a}, \frac{b}{a} - n \\ 1+\frac{b}{a}, 1+\frac{b}{a}, 1+\frac{b}{a} \end{matrix} \middle| 1 \right] \\ &= \frac{n!}{2a^3 \left(\frac{b}{a}\right)_{n+1}} \left[\left(H^{(1)}_{n+\frac{b}{a}} - H^{(1)}_{\frac{b}{a}-1} \right)^2 + H^{(2)}_{n+\frac{b}{a}} - H^{(2)}_{\frac{b}{a}-1} \right], \end{aligned}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(ak+b)^4}=\frac{-1}{6a^4}\int_0^1 x^{\left(\frac{b}{a}-1\right)}(1-x)^n\ln^3(x)\,dx$$
$$=\frac{1}{b^4}\;{}_5F_4\left[\begin{array}{c}\frac{b}{a},\frac{b}{a},\frac{b}{a},\frac{b}{a}-n\\ 1+\frac{b}{a},1+\frac{b}{a},1+\frac{b}{a},1+\frac{b}{a}\end{array}\middle|1\right]$$
$$=\frac{n!}{6a^4\ \left(\frac{b}{a}\right)_{n+1}}\left[\begin{array}{c}\left(H^{(1)}_{n+\frac{b}{a}}-H^{(1)}_{\frac{b}{a}-1}\right)^3+3\left(H^{(1)}_{n+\frac{b}{a}}-H^{(1)}_{\frac{b}{a}-1}\right)\left(H^{(2)}_{n+\frac{b}{a}}-H^{(2)}_{\frac{b}{a}-1}\right)\\ +2\left(H^{(3)}_{n+\frac{b}{a}}-H^{(3)}_{\frac{b}{a}-1}\right).\end{array}\right].$$

Note that the inversion formula states that

$$g(n)=\sum_k(-1)^k\binom{n}{k}f(k)\ \text{ if and only if }f(n)=\sum_k(-1)^k\binom{n}{k}g(k)$$

and therefore we may write, from (2.5) that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k!}{\left(\frac{b}{a}\right)_{k+1}}\left(H^{(1)}_{k+\frac{b}{a}}-H^{(1)}_{\frac{b}{a}-1}\right)=\frac{a^2}{(an+b)^2}.$$

Some interesting extensions of Theorem 1 are as follows.

Theorem 2. *Let p be a positive integer. Then for real positive numbers a and b we have*

$$\sum_{k=0}^{n}k\binom{n}{k}\frac{(-1)^{k+1}}{(ak+b)^p}=\frac{n(-1)^{p-1}}{a^p\Gamma(p)}\int_0^1 x^{\frac{b}{a}}(1-x)^{n-1}\ \ln^{p-1}(x)\,dx \tag{2.6}$$
$$=\frac{n}{(a+b)^p}\quad{}_{p+1}F_p\left[\begin{array}{c}\overbrace{\frac{b}{a}+1,\ldots,\frac{b}{a}+1}^{p-terms},1-n\\ \underbrace{2+\frac{b}{a},\ldots,2+\frac{b}{a}}_{p-terms}\end{array}\middle|1\right]$$

and

$$\sum_{k=0}^{n}k^2\binom{n}{k}\frac{(-1)^{k+1}}{(ak+b)^p}=\frac{n(-1)^{p-1}}{a^p\Gamma(p)}\int_0^1 x^{\frac{b}{a}}(1-x)^{n-2}\ (1-nx)\ln^{p-1}(x)\,dx \tag{2.7}$$
$$=\frac{n}{(a+b)^p}\quad{}_{p+2}F_{p+1}\left[\begin{array}{c}\overbrace{\frac{b}{a}+1,\ldots,\frac{b}{a}+1}^{p-terms},2,1-n\\ \underbrace{2+\frac{b}{a},\ldots,2+\frac{b}{a}}_{p-terms},\ 1\end{array}\middle|1\right].$$

Proof. Using Lemma 1 and interchanging the sum and integral, we may write

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{k}{(ak+b)^p}\\
&\quad=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{k\ (-1)^{p-1}}{a^p\ \Gamma(p)}\int_0^1 x^{\frac{b}{a}-1}\ x^k\ln^{p-1}(x)\ dx\\
&\quad=\frac{(-1)^{p-1}}{a^p\Gamma(p)}\int_0^1 x^{\frac{b}{a}-1}\ln^{p-1}(x)\sum_{k=1}^{n}k\binom{n}{k}(-x)^k\ dx\\
&\quad=\frac{(-1)^{p-1}}{a^p\Gamma(p)}\int_0^1 x^{\frac{b}{a}-1}\ln^{p-1}(x)\ \ n\ x\ \ (1-x)^{n-1}\ dx\\
&\quad=\frac{n\,(-1)^{p-1}}{a^p\Gamma(p)}\int_0^1 x^{\frac{b}{a}}\ln^{p-1}(x)\,(1-x)^{n-1}\ dx.
\end{aligned}$$

Therefore, the integral in (2.6) is attained. The integral (2.7) can be proved in the same way. The hypergeometric representations can be attained by considering the ratio of binomial terms in the sums (2.6) and (2.7). □

Example 3. *For $p=3$ we have*

$$\begin{aligned}
\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k\,k}{(ak+b)^3}&=\frac{n}{2a^3}\int_0^1 x^{\left(\frac{b}{a}-1\right)}(1-x)^n\ln^2(x)\ dx\\
&=\frac{n}{(a+b)^3}\ {}_4F_3\left[\left.\begin{matrix}\frac{b}{a},\frac{b}{a},\frac{b}{a}-n\\ 2+\frac{b}{a},2+\frac{b}{a},2+\frac{b}{a}\end{matrix}\right|1\right]\\
&=\frac{bn!}{2a^4\ \left(\frac{b}{a}\right)_{n+1}}\left[\left(H^{(1)}_{n+\frac{b}{a}}-H^{(1)}_{\frac{b}{a}}\right)^2-\left(H^{(2)}_{n+\frac{b}{a}}-H^{(2)}_{\frac{b}{a}}\right)\right].
\end{aligned}$$

The next theorem follows.

Theorem 3. *Let p be a positive integer. Then for real positive numbers a and b we have*

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\ (ak+b)^p}\\
&\quad=\frac{H_n}{b^p}-\frac{ap}{b^{p+1}}+\frac{1}{b^p\Gamma(p)}\int_0^1\frac{(1-x)^n}{x}\Gamma\left(p,-\frac{b}{a}\ln(x)\right)\ dx \qquad (2.8)\\
&\quad=\frac{n}{(a+b)^p}\ \ {}_{p+3}F_{p+2}\left[\left.\begin{matrix}\overbrace{\frac{b}{a}+1,\ldots,\frac{b}{a}+1}^{p-terms},1,1,1-n\\ \underbrace{2+\frac{b}{a},\ldots,2+\frac{b}{a},}_{p-terms}\ 2,2\end{matrix}\right|1\right],
\end{aligned}$$

where the incomplete Gamma function is given by

$$\Gamma(p,z)=\int_z^\infty\ t^{p-1}\exp(-t)\ dt.$$

Proof. By expansion,

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\ \ (ak+b)^{p}}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\ b^{p}}$$
$$-a\sum_{k=1}^{n}\ (-1)^{k+1}\left(\begin{array}{c}n\\k\end{array}\right)\sum_{r=1}^{p}\ \frac{(-1)^{k+1}}{b^{p+1-r}\,(ak+b)^{r}}$$
$$=\frac{H_n}{b^p}-a\sum_{r=1}^{p}\frac{1}{b^{p+1-r}}\left(\frac{1}{b^r}+\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{(ak+b)^{r}}\right)$$
$$=\frac{H_n}{b^p}-a\sum_{r=1}^{p}\frac{1}{b^{p+1}}-a\sum_{r=1}^{p}\frac{1}{b^{p+1-r}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{(ak+b)^{r}}.$$

The last term follows directly from Theorem 1, (2.3), so that

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\,(ak+b)^{p}}=\frac{H_n}{b^p}-a\sum_{r=1}^{p}\frac{1}{b^{p+1}}$$
$$-\frac{a}{b^{p+1}}\sum_{r=1}^{p}\frac{b^r\,(-1)^{r-1}}{a^r\Gamma\,(r)}\int_0^1 x^{\frac{b}{a}}\frac{(1-x)^n}{x}\ln^{r-1}(x)\ dx$$
$$=\frac{H_n}{b^p}-\frac{ap}{b^{p+1}}-\frac{a}{b^{p+1}}\sum_{r=1}^{p}\frac{b^r\,(-1)^{r-1}}{a^r\ \Gamma\,(r)}\int_0^1 x^{\frac{b}{a}}\frac{(1-x)^n}{x}\ln^{r-1}(x)\ dx$$
$$=\frac{H_n}{b^p}-\frac{ap}{b^{p+1}}-\frac{a}{b^{p+1}}\int_0^1 x^{\frac{b}{a}}\frac{(1-x)^n}{x}\frac{b}{a}\sum_{r=1}^{p}\frac{\left(-\frac{b}{a}\ln(x)\right)^{r-1}}{\Gamma\,(r)}dx$$
$$=\frac{H_n}{b^p}-\frac{ap}{b^{p+1}}-\frac{a}{b^{p+1}}\int_0^1 x^{\frac{b}{a}}\frac{(1-x)^n}{x}\frac{b}{a}\ \frac{x^{-\frac{b}{a}}\Gamma\left(p,-\frac{b}{a}\ln(x)\right)}{\Gamma\,(p)}dx$$

and (2.8) follows. The hypergeometric representation follows from the ratio of the binomial terms. □

Example 4. *First we can note that, from Theorem 3, when p is a positive integer we have*

$$\Gamma\,(p,z)=\int_z^\infty\ t^{p-1}\exp(-t)\ \ dt=(p-1)!\exp(-z)\sum_{j=0}^{p-1}\frac{z^j}{j!}$$
$$=(p-1)!\exp(-z)\ \ \Xi_{p-1}(z)$$

and $\Xi_{n-1}(z)$ is denoted as the exponential sum function. For $p=2$, $\Gamma\,(2,-m\ln x)=x^m\,(1-m\ln x)$ and hence,

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\ \ (ak+b)^{2}}=\frac{n!}{b\left(\frac{b}{a}\right)_{n+1}}\left(\frac{2}{b}+\frac{1}{a}\left(H^{(1)}_{n+\frac{b}{a}}-H^{(1)}_{\frac{b}{a}}\right)\right)+\frac{H_n}{b^2}-\frac{2a}{b^3}\tag{2.9}$$
$$=\frac{n}{(a+b)^2}\ \ {}_5F_4\left[\begin{array}{c}1+\frac{b}{a},1+\frac{b}{a},1,1,1-n\\2+\frac{b}{a},2+\frac{b}{a},2,2\end{array}\middle|\,1\right]$$
$$=\frac{H_n}{b^2}-\frac{ap}{b^3}+\frac{1}{b^2}\int_0^1\ (1-x)^n\ \ x^{\frac{b}{a}-1}\left(1-\frac{b}{a}\ln x\right)\ dx.$$

Remark 1. *Infinite versions of some of the identities and their modifications have been investigated. In* [14]

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,(an+b)^{m+1}}=\frac{1}{b^{m+1}}\left\{H^{(1)}_{\frac{b}{2a}}-H^{(1)}_{\frac{b}{a}}\right\}+\sum_{p=2}^{m+1}\frac{a}{b^{m+2}}\left(\frac{b}{2a}\right)^p\left(H^{(p)}_{\frac{b}{2a}}-H^{(p)}_{\frac{b}{2a}-\frac{1}{2}}\right)$$

and [16]

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{m+1}\,(an+1)}=\sum_{s=0}^{m-1}\left(2^{s-m}-1\right)\zeta\,(m+1-s)+(-1)^m\,a^m\left(H^{(1)}_{\frac{1}{a}}-H^{(1)}_{\frac{1}{2a}}\right).$$

Remark 2. *Although the focus of this paper is on integral representations of sums it should be noted that the sums in (2.2), (2.6), and (2.8) can be solved by recurrence relations* [12]. *The emphasis in this paper is in giving another tool, integral representations for the relevant sums which enable the study of convexity properties* [19]. *We give some examples of recurrence relations. For*

$$S_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(2k+1)}$$

then

$$(2n+3)\,S_{n+1}-2\,(n+1)\,S_n=0,\;\;S_0=1,$$

and solving produces (1.1). For

$$W_n=\sum_{k=0}^{n}\;(-1)^k\binom{n}{k}\frac{1}{k\,(ak+b)^2}$$

we obtain

$$\left(a^2\,(n+3)^2+b\,(2an+6a+b)\right)W_{n+3}-\begin{pmatrix}a^2\left(3n^2+15n+19\right)\\ +b\,(4an+10a+b)\end{pmatrix}W_{n+2}$$

$$+\begin{pmatrix}3a^2\,(n+2)^2\\ +2a\,(6an+bn+2b)\end{pmatrix}W_{n+1}-a^2\,(n+2)\,(n+1)\,W_n=\frac{1}{n+3},\;\;W_1=\frac{1}{(a+b)^2},$$

$$W_2=\frac{(5a+3b)\,(3a+b)}{2\,(a+b)^2\,(2a+b)^2},\;\;W_3=\frac{575a^4+888a^3b+494a^2b^2+120ab^3+11b^4}{6\,(a+b)^2\,(a+b)^2\,(2a+b)^2\,(3a+b)^2}$$

and solving produces the solution (2.9).

3. Acknowledgment

The author is grateful to an anonymous referee for suggestions leading to a better presentation of this paper.

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MSC2010: 05A10, 11B65, 05A19, 33C20

Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia
E-mail address: anthony.sofo@vu.edu.au