A RESULT ABOUT CYCLES IN DUCCI SEQUENCES

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ABSTRACT. We prove that for any $k \in \mathbb{N}$, k not a power of two, there are cyclic vectors of length k which are not the concatenation of two or more copies of a vector of smaller length. As an application of this, we give a new proof of the fact that the period of a Ducci sequence can be any positive integer with the exception of the powers of 2 greater than 1.

1. Introduction to Ducci Sequences

Let $k \in \mathbb{N}$ and let $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$. We define a map $T : \mathbb{N}^k \to \mathbb{N}^k$ by

$$T(\vec{x}) = T(a_0, a_1, \ldots, a_{k-1}) = (|a_0 - a_1|, |a_1 - a_2|, \ldots, |a_{k-1} - a_0|).$$

The sequence $(T^n(\vec{x}))_{n\in\mathbb{N}}$ generated by the iterations of T is called a *Ducci sequence*. Ducci sequences have been extensively studied and often rediscovered. We state a few well-known facts which will be used in this paper.

Let $\vec{x} = (a_0, a_1, \ldots a_{k-1}) \in \mathbb{N}^k$. If there exists $a \in \mathbb{N}$ such that for every $0 \le i \le k$, $a_i \in \{0, a\}$, we will say that $\vec{x}$ is a *simple* vector. A well-known result states that for every $\vec{x} \in \mathbb{N}^k$, there exists $n \in \mathbb{N}$ such that $T^n(\vec{x})$ is simple (see [5] for example). Since there are only finitely many vectors of length k with components in $\{0, a\}$, the iterations must eventually repeat. In other words, every Ducci sequence is ultimately cyclic. We derive another important consequence of this fact.

Let $\vec{x} = (a_0, a_1, \ldots, a_{k-1})$ be a simple vector with $a_i \in \{0, a\}$ for every $0 \le i \le k-1$. We can rewrite $\vec{x}$ as $(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1})$, where $\epsilon_i = 1$ if $a_i = a$ and 0 otherwise. Notice that for every $k \in \mathbb{N}$, $T^k(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1}) = a \cdot T^k(\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$. This simple remark implies that in order to study cycles in Ducci sequences, we can restrict our attention to vectors with components in $\{0, 1\}$.

Finally, note that when $a_i \in \{0, 1\}$, the operation $|a_i - a_{i+1}|$ is equivalent to $a_i + a_{i+1} \pmod 2$. Consequently, we will study the map $T : \mathbb{Z}_2^k \to \mathbb{Z}_2^k$ defined by

$$T(a_0, a_1, \ldots, a_{k-1}) = (a_0 + a_1, a_1 + a_2, \ldots, a_{k-1} + a_0).$$

We will call the sequences generated by iterating this map *Ducci sequences over* $\mathbb{Z}_2$. If for some vector $\vec{x} \in \mathbb{N}^k$, there exists an integer p such that $T^p(\vec{x}) = \vec{x}$, and p is minimal with this property, we will say that $\vec{x}$ has period p. The discussion above implies that in order to study the periods of the Ducci sequences it is sufficient to do so over $\mathbb{Z}_2$. We will do so in the remainder of this paper.

2. Proof of the Main Result

In order to simplify the notation, the indices of the components of any vector $\vec{x} \in \mathbb{Z}_2^k$ will be written modulo k so that, for example, $a_k = a_0$ and $a_{k+1} = a_1$.

If $T^m(\vec{x}) = \vec{x}$ for some integer m, we will say that $\vec{x}$ is *cyclic*. If m is the smallest such integer, we say that $\vec{x}$ is m-cyclic, or as defined earlier, has period m. If for some integer n, we have $T^n(\vec{x}) = \vec{y}$ and $\vec{y}$ is m-cyclic, we say that $\vec{y}$ *belongs to the cycle generated by* $\vec{x}$ and we write $c(\vec{x}) = m$. In other words, $c(\vec{x})$ is the length of the cycle that the iterations of $\vec{x}$ will eventually reach. If $T^m(\vec{x}) = \vec{0}$ for some integer m, we say that $\vec{x}$ is *nilpotent*.

Given $k, l \in \mathbb{N}$ and two vectors $\vec{x} = (x_0, x_1, \ldots, x_k)$ and $\vec{y} = (y_0, y_1, \ldots, y_l)$ we denote by $\vec{x} \vee \vec{y}$ their concatenation $(x_0, x_1, \ldots, x_k, y_1, y_2, \ldots, y_l)$. For $m > 1$, we write $\vec{x} \vee \vec{x} \ldots \vee \vec{x} = \vee^{(m)}\vec{x}$, the concatenation of $\vec{x}$ with itself m times and by convention $\vee^{(1)}\vec{x} = \vec{x}$.

It is easy to see that for any $\vec{x} \in \mathbb{Z}_2^k$ (or in $\mathbb{N}^k$) and for any positive integers n, m the following relation holds:

$$T^n(\vee^{(m)}\vec{x}) = \vee^{(m)}T^n(\vec{x}). \tag{2.1}$$

In particular, $T^n(\vec{x}) = \vec{0}$ implies $T^n(\vee^{(m)}\vec{x}) = \vec{0}$ for every m.

A fundamental theorem states that if $k = 2^l$ for some l, then any $\vec{x} \in \mathbb{Z}_2^k$ is nilpotent (see for example [3]). Together with (2.1), this implies that for any $m \in \mathbb{N}$ and any $\vec{x} \in \mathbb{Z}_2^{2^l}$, the vector $\vee^{(m)}\vec{x}$ is nilpotent. In [1] a converse to this statement was proved, thus showing the following proposition.

Proposition 2.1. *Let $\vec{x} \in \mathbb{Z}_2^k$. The vector $\vec{x}$ is nilpotent if and only if there exist $l, m \in \mathbb{N}$ and $\vec{y} \in \mathbb{Z}_2^{2^l}$ such that $\vec{x} = \vee^{(m)}\vec{y}$.*

The equality (2.1) also allows us to construct new cyclic vectors: if $\vec{x}$ is cyclic of period p, so is $\vee^{(m)}\vec{x}$ for any $m > 1$. The main result of this paper is that for any $k \in \mathbb{N}$, k not a power of two, there are cyclic vectors in $\mathbb{Z}_2^k$ (and thus also in $\mathbb{N}^k$) that are not the concatenation of two or more copies of a smaller vector. So in essence, for every k which is not a power of 2, $\mathbb{Z}_2^k$ has "original" cyclic vectors. Note that this result is obvious if $k > 2$ is prime and not true if k is a power of 2 since in this case only $\vec{0}$ is cyclic. We will need two simple lemmas.

Lemma 2.2. *Let $k \in \mathbb{N}$ and $\vec{x}$ be any element of $\mathbb{Z}_2^k$. There exists a unique $\vec{y}$ in the cycle generated by $\vec{x}$ and a unique nilpotent $\vec{z}$ such that $\vec{x} = \vec{y} + \vec{z}$.*

Proof. First we show the existence. If $\vec{x}$ belongs to the cycle generated by itself, take $\vec{z} = \vec{0}$. Otherwise, choose n such that $\vec{c} = T^n(\vec{x})$ is cyclic. Denote by m the period of $\vec{c}$ and choose l such that $lm > n$. Define $\vec{z} = \vec{x} + T^{lm-n}(\vec{c})$. The linearity of T over $\mathbb{Z}_2^k$ implies that $T^n(\vec{z}) = T^n(\vec{x}) + T^{n+lm-n}(\vec{c}) = \vec{c} + T^{lm}(\vec{c}) = \vec{c} + \vec{c} = \vec{0}$. In other words, $\vec{z}$ is nilpotent and we can write $\vec{x} = T^{lm-n}(\vec{c}) + \vec{z}$.

To show uniqueness, suppose $\vec{x} = \vec{y} + \vec{z} = \vec{y}\,' + \vec{z}\,'$ where $\vec{y}$ and $\vec{y}\,'$ both belong to the cycle generated by $\vec{x}$ and $\vec{z}, \vec{z}\,'$ are both nilpotent. Let n be an integer such that $T^n(\vec{z}) = T^n(\vec{z}\,') = \vec{0}$. Then

$$\begin{aligned} &T^n(\vec{y} + \vec{z}) = T^n(\vec{y}\,' + \vec{z}\,') \\ &\Rightarrow T^n(\vec{y}) + T^n(\vec{z}) = T^n(\vec{y}\,') + T^n(\vec{z}\,') \\ &\Rightarrow T^n(\vec{y}) = T^n(\vec{y}\,') \end{aligned}$$

But since both $\vec{y}$ and $\vec{y}\,'$ belong to the same cycle we must have $\vec{y} = \vec{y}\,'$. It follows that $\vec{z} = \vec{z}\,'$ □

Define Nil_k to be the set of nilpotent vectors in $\mathbb{Z}_2^k$. Lemma 2.2 immediately implies the following.

Lemma 2.3. *Let $\vec{x} \in \mathbb{Z}_2^k$. Exactly one element in the set $\{\vec{x} + \vec{z},\ \vec{z} \in Nil_k\}$ is cyclic.*

Proof. Suppose that there are two different $\vec{z}_1$ and $\vec{z}_2$ in Nil_k such that $\vec{x}+\vec{z}_1=\vec{y}_1$ and $\vec{x}+\vec{z}_2=\vec{y}_2$ are both cyclic. Then we can rewrite these equalities as $\vec{x}=\vec{y}_1+\vec{z}_1$ and $\vec{x}=\vec{y}_2+\vec{z}_2$, contradicting the previous lemma. □

Define C_k to be the set of cyclic elements of $\mathbb{Z}_2^k$ and

$$S_k=\{\vec{x}\in C_k: \text{ there exists } \vec{y} \text{ such that } \vec{x}=\vee^{(n)}\vec{y} \text{ for some } n\geq 2\}.$$

Theorem 2.4. *For every positive integer k, k not a power of 2, there is at least one cyclic vector of length k which is not the concatenation of two or more copies of a smaller vector.*

Proof. Set $k=2^l m$ for some odd number $m\geq 3$. The statement is equivalent to showing that the set $C_k\setminus S_k$ is non-empty. There are exactly 2^{2^l} vectors in $\mathbb{Z}_2^k$ that can be obtained by concatenating vectors of length a power of 2. Proposition 2.1 implies $|Nil_k|=2^{2^l}$. Using Lemma 2.3 we obtain $|C_k||Nil_k|=|\mathbb{Z}_2^k|$ or $|C_k|2^{2^l}=2^{2^l m}$. Consequently,

$$|C_k|=2^{2^l(m-1)}. \tag{2.2}$$

Notice that if $\vec{x}=\vee^{(n)}\vec{y}$ for $n\geq 2$ then $\vec{y}$ can be of length at most $k/2$. Thus, the values of the first $\lfloor k/2\rfloor$ components of $\vec{x}$ determine $\vec{x}$ entirely. Since the vector $(1,1,\ldots,1)$ is not in C_k, we have the following upper bound on the size of S_k:

$$|S_k|\leq 2^{\lfloor k/2\rfloor}-1\leq 2^{2^{l-1}m}-1.$$

The last inequality in conjunction with (2.2) gives us the following lower bound on the size of $C_k\setminus S_k$

$$|C_k-S_k|\geq 2^{2^l(m-1)}-(2^{2^{l-1}m}-1)\geq 1, \tag{2.3}$$

since $m\geq 3$, concluding the proof. □

The equality (2.3) in the above proof was first proved by Ludington-Young in [7] and later in [2] by Brown and Merzel. If 2_k denotes the highest power of 2 dividing k, then it can be restated as the following corollary.

Corollary 2.5 (Young's Theorem). *For every $k\in\mathbb{N}$, the number of cyclic vectors is 2^{k-2_k}.*

If we denote by R the rotation of components defined by $R((a_0,a_1,\ldots,a_{k-1}))=(a_1,a_2,\ldots,a_0)$, then $R(\vec{x})$ is cyclic whenever $\vec{x}$ is cyclic. Note also that if $\vec{x}$ is not the concatenation of two or more copies of a shorter vector, then for any $n\in\mathbb{N}$, $R^n(\vec{x})\neq\vec{x}$. Consequently, we actually have $|C_k\setminus S_k|\geq k$, strengthening Theorem 2.4.

Corollary 2.6. *For any k not a power of 2, $|C_k\setminus S_k|\geq k$.*

3. An Application of Theorem 2.4

In this section, we begin by proving that the period of a Ducci sequence cannot be a power of 2 greater than 1 (the null vector has period 1).

Proposition 3.1. *For every integer $m\geq 1$ and every $\vec{x}$ in $\mathbb{Z}_2^k$, $c(\vec{x})\neq 2^m$.*

Proof. Suppose $\vec{x}$ is a cyclic vector such that $c(\vec{x})=2^m$ for some positive integer m. Consider the vector $\vec{x}_1=\vec{x}+T^{2^{m-1}}(\vec{x})$. It cannot be that $\vec{x}_1=\vec{0}$, otherwise $\vec{x}=T^{2^{m-1}}(\vec{x})$ and $c(\vec{x})\leq 2^{m-1}$, contradicting our assumption. Also notice that

$$T^{2^{m-1}}(\vec{x}_1)=T^{2^{m-1}}(\vec{x}+T^{2^{m-1}}(\vec{x}))=T^{2^{m-1}}(\vec{x})+T^{2^m}(\vec{x})=T^{2^{m-1}}(\vec{x})+\vec{x}=\vec{x}_1.$$

Thus from $\vec{x}$ we constructed a non-zero cyclic vector $\vec{x}_1$ whose period $c(\vec{x}_1)$ must be a divisor of 2^{m-1}. If we repeat the procedure with $\vec{x}_1$, we obtain a cyclic vector $\vec{x}_2$ whose period divides 2^{m-2} and $\vec{x}_2 \neq \vec{0}$. By repeating the process m times, we create a non-null cyclic vector $\vec{x}_m$ with $c(\vec{x}_m) = 1$, a contradiction. □

We use Theorem 2.4 to show that any positive integer which is a power of 2 is the period of a Ducci sequence. This fact is also the consequence of a more general result proved in [2]. Related results were also proved earlier in [7]. Both [7, 2] exploit a form of duality between the "rows" and "columns" of Ducci sequences. We use the same idea in the following proof.

Theorem 3.2. *Let m be an integer. If $m \neq 2^t$ for any $t \geq 2$, then there exist k and a vector $\vec{x}$ in $\mathbb{Z}_2^k$ such that $c(\vec{x}) = m$.*

Proof. Let $k = 2^l q$ for some integers l, q where $q \geq 3$ is odd. Using Theorem 2.4, let $\vec{x} = \{x_0, x_1, \ldots, x_{k-1}\} \in C_k \setminus S_k$ and form the matrix $M = [a_{i,j}]$, $1 \leq i \leq c(\vec{x})$, $0 \leq j \leq k-1$ where $a_{i,j}$ is the j component of $T^i(\vec{x})$. By construction

$$a_{i,j} + a_{i,j+1} = a_{i+1,j}$$

or equivalently

$$a_{i+1,j} + a_{i,j} = a_{i,j+1}. \tag{3.1}$$

Consider for $0 \leq i \leq k-1$ the vectors $\vec{C}_i = (a_{c(\vec{x}),i}, a_{c(\vec{x})-1,i}, \ldots, a_{1,i})$, the transposition of the $i+1$ column of M. Note that the role of the indices is now reversed. In particular i now represent the row. By (3.1) we have $T(\vec{C}_i) = \vec{C}_{i+1}$, where as usual the addition of the index is taken modulo k. In particular

$$T^k(\vec{C}_1) = \vec{C}_1. \tag{3.2}$$

We claim that $c(\vec{C}_1) = k$. By (3.2) we have $c(\vec{C}_1)|k$. Suppose that $c(\vec{C}_1) = t$ is a proper divisor of k. Then $T^t(\vec{C}_1) = \vec{C}_1$ implying $x_0 = x_t$. Similarly, since then $T^t(C_i) = C_i$ for every $0 \leq i \leq k-1$ we obtain in general

$$x_i = x_{i+t}.$$

But then $\vec{x} = (x_0, x_1, \ldots, x_{t-1}, x_0, \ldots, x_{t-1}, \ldots, x_0, \ldots, x_{t-1})$, contradicting the fact that $\vec{x} \notin S_k$. □

References

[1] C. Avart, *A characterization of converging Ducci sequences over $\mathbb{Z}_2$*, The Fibonacci Quarterly, **49.2** (2011), 155–157.

[2] R. Brown and J. L. Merzel, *The number of Ducci sequences with given period*, The Fibonacci Quarterly, **45.2** (2007), 115-121.

[3] R. Brown and J. L. Merzel, *Limiting behaviour in Ducci sequences*, Periodica. Math. Hungarica, **47** (2003), 45–50.

[4] A. Ehrlich, *Periods in Ducci's n-number game of differences*, The Fibonacci Quarterly, **26.2** (1990), 302–305.

[5] A. Ehrlich, *Columns of differences*, Mathematics Teaching, **79** (1977), 42–45.

[6] H. Glaser and G. Schöffl, *Ducci-sequences and Pascal's triangle*, The Fibonacci Quarterly, **33.4** (1995), 313–324.

[7] A. Ludington Furno, *Cycles of differences of integers*, J. Number Theory, **13** (1981), 255–261.

[8] A. Ludington-Young, *Length of the n-number game*, The Fibonacci Quarterly, **28.3** (1990), 259–265.
[9] A. Ludington-Young, *Even Ducci-sequences*, The Fibonacci Quarterly, **37.2** (1999), 145–153.

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