BINARY WORDS, n-COLOR COMPOSITIONS AND BISECTION OF THE FIBONACCI NUMBERS

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Abstract. An n-color composition of n is a composition of n where a part k has k possible colors. It is known that the number of n-color compositions of n is F_{2n} (the $2n$th Fibonacci numbers). Among other objects, F_{2n} also counts the number of binary words with exactly $n-1$ strictly increasing runs and the number of $\{0,1,2\}$ strings of length $n-1$ excluding the subword 12. In this note, we show bijections between n-color compositions and these objects. In particular, the bijection between the n-color compositions and the binary words with $n-1$ increasing substrings generalizes the classic bijection between compositions and binary words of length $n-1$. We also comment on the potential applications of these findings.

1. Introduction

A composition of a positive integer n is an ordered t-tuple of positive integers $\{c_1, c_2, \ldots, c_t\}$ (we call each c_i a "part" of this composition) such that

$$c_1 + c_2 + \cdots + c_t = n.$$

It is well-known that the number of compositions of n is 2^{n-1} [8]. For instance, the 8 compositions of 4 are listed below:

$$4, 3+1, 2+2, 1+3, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1.$$

A composition can be naturally represented as a tiling of a $1 \times n$ board with tiles of dimension $1 \times c_i$ where $1 \leq i \leq t$. For example, the composition $2+3+1 = 6$ can be represented as Figure 1.

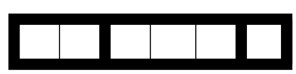

Figure 1. Tiling representation of $2+3+1=6$.

The classical bijection that maps a composition of n to a binary word of length $n-1$ is due to MacMahon [8]. One can also see, for instance, [5] for details. In terms of the tiling representation, one could consider the internal vertical lines in Figure 1 and map a vertical line to a 0 if it is bold and otherwise to a 1. The above composition of 6 is then mapped to a binary word

$$10110 \tag{1.1}$$

of length 5.

An "n-color composition" of n is a composition where a part k has one of k possible colors [1]. This can be conveniently denoted by a subscript for each part (i.e., k_i denotes a part k with color i, where $1 \leq i \leq k$). For instance, in the composition $2+3+1$, the part 2 has two

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possible colors, the part 3 has three possible colors, the part 1 has only one possible color. All the n-color compositions with these parts (in this order) are shown below:

$$2_1 + 1_1 + 3_1, 2_1 + 1_1 + 3_2, 2_1 + 1_1 + 3_3, 2_2 + 1_1 + 3_1, 2_2 + 1_1 + 3_2, 2_2 + 1_1 + 3_3. \qquad (1.2)$$

The tiling representation can be naturally extended for n-color compositions by marking one of the k squares of an $1 \times k$ tile with a dot, called the "spotted tiling" [7]. For example, the n-color composition $2_2 + 3_2 + 1_1 = 6$ can be represented by Figure 2.

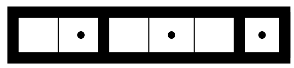

FIGURE 2. Tiling representation of $2_2 + 3_2 + 1_1 = 6$.

Many interesting studies and questions followed from the introduction of the n-color compositions. See, for instance, [2, 4] and the references therein for some related topics. Even before the formal introduction of this concept, "weighted compositions" were considered [6] (in the setting of n-color compositions, the weight of a part is simply the number of colors). When introducing this concept, Agarwal [1] also asked for the analogue of MacMahon's zig-zag graph [9] as the conjugation of a composition.

It is known that the number of n-color compositions of n is F_{2n} [1], i.e., bisection of Fibonacci sequence (A001906 in [10]). In [7], an interesting bijection was established between the n-color compositions of n and compositions of $2n$ with odd parts.

In view of the aforementioned classical one-to-one correspondence between the compositions and the binary words, it is interesting to see that the "number of binary words with exactly $n-1$ strictly increasing runs" (A119900 in [10]) is also F_{2n}. Formally speaking, a strictly "increasing run" is a maximal subsequence of consecutive terms that is strictly increasing. Evidently each "strictly increasing run" is either a 0, 1, or 01. As an example, the binary word "100111101011100" can be written as

$$|1|0|01|1|1|1|01|01|1|1|0|0|$$

where strictly increasing runs are separated by $|$'s.

In Section 2, we will see a bijection between the n-color compositions and such binary words, offering a combinatorial proof for the following fact.

Proposition 1.1. *The number of n-color compositions of n is the same as the number of binary words with exactly $n-1$ strictly increasing runs.*

Essentially the same bijection also leads to the following proposition.

Proposition 1.2. *The number of n-color compositions of n is the same as the number of ternary strings of length $n-1$ without 12.*

The second set of objects in Proposition 1.2 are also counted by A001906 in [10]. See, for instance [3]. In fact, we will see that a one-to-one correspondence between the binary words and ternary strings follows naturally.

Proposition 1.1 provides a generalization of the bijection between compositions and binary words. And indeed from any binary string, one can generate a corresponding n-color composition following the bijection. We comment on the potential applications of these results in Section 3.

2. The Bijections

We start with the bijection between the n-color compositions of n and binary words with $n-1$ strictly increasing runs. This bijection not only generalizes the classical bijection between compositions and binary words, but also shows some potential in dealing with different questions such as defining the conjugation or obtaining bijections between n-color compositions.

2.1. Between n-color compositions and binary words.

The bijection is presented here in an algorithmic process. From the spotted tiling representation of an n-color composition, we start from the leftmost tile. In every step (except the last) we decrease the number of "squares" by 1 and generate a strictly increasing run. This is described below and illustrated by Figure 3.

- If the leftmost part is 1_1 (i.e., a tile with one square that is marked), we remove this tile and add $|01|$ to the binary word;
- If the leftmost part is k_i for some $k > 1$ and $1 < i \leq k$ (i.e., the leftmost tile is of size greater than 1 and the marked square is not the first (leftmost) one), we remove the first square (hence the new first part will be $(k-1)_{i-1}$) and add $|1|$ to the binary word;
- If the leftmost part is k_1 for some $k > 1$ (i.e., the leftmost tile is of size greater than 1 and the marked square is the first one), we remove the first square and mark the next one (hence the new first part will be $(k-1)_1$), adding $|0|$ to the binary word.

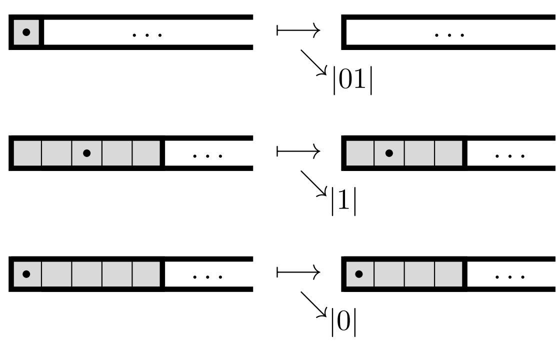

Figure 3. Generate a binary word from an n-color composition.

Following this process, we will always have a tile of size 1 with the square marked in the end. We simply ignore this last square. This is shown by Figure 4, when there are two squares left before the last step.

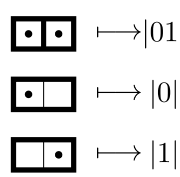

Figure 4. Three possibilities for the end of the process.

It is not difficult to see that the operations defined this way maps the tiling representation of an n-color composition of n to a binary word with $n-1$ strictly increasing runs. For example, the n-color composition

$$3_2 + 4_4 + 1_1 + 5_3 = 13$$

has a tiling representation as in Figure 5.

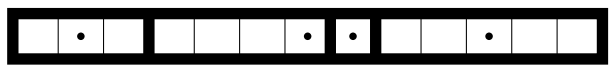

FIGURE 5. Tiling representation of $3_2 + 4_4 + 1_1 + 5_3 = 13$.

Under the operations defined above, this representation is mapped to the binary word

$$|1|0|01|1|1|1|01|01|1|1|0|0|. \tag{2.1}$$

Instead of providing a formal proof for this bijection, we show in detail how one can reverse the process and achieve an n-color composition of 13 from (2.1).

Starting from the right-hand side of the binary word, we have $|0|$ corresponding to the second operation in Figure 4. Then we have another $|0|$ corresponding to the third operation in Figure 3 and the process continues this way. This is illustrated in Figure 6.

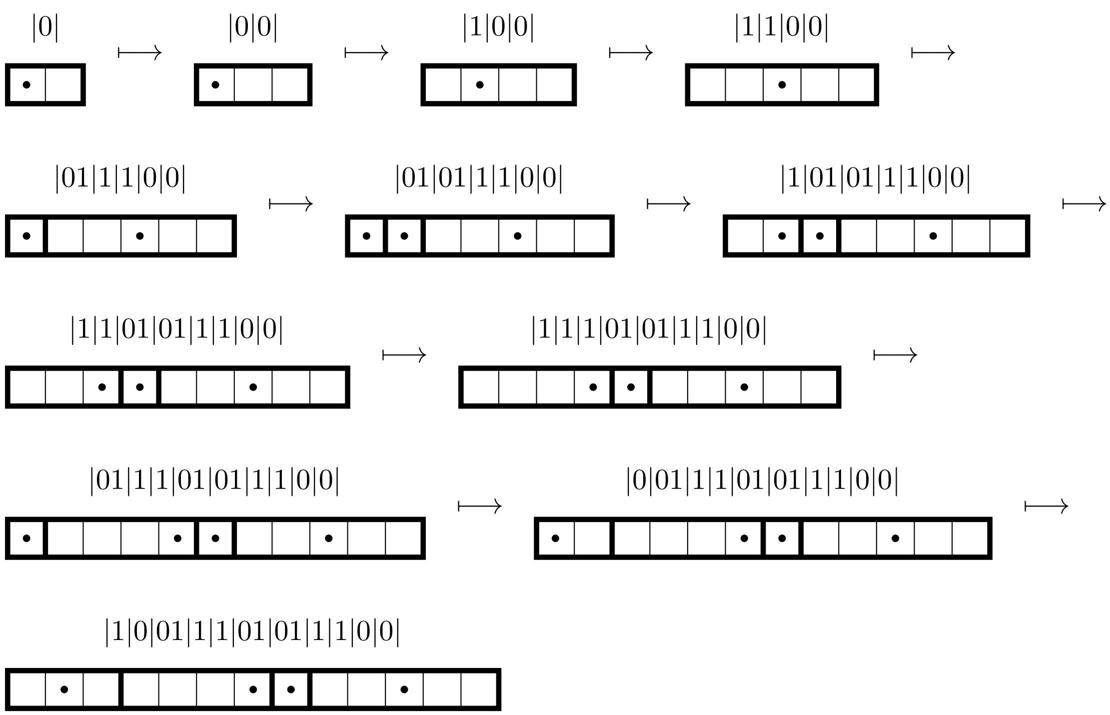

FIGURE 6. Generating an n-color composition from a binary word.

2.2. **Between n-color compositions and ternary strings without 12.** The process is similar to that in the previous subsection, illustrated in Figure 7. Note that any string generated from an n-color composition will avoid the pattern "12". We omit the details.

Similar to before, we ignore the very end of the tiling. Again, Figure 8 shows the situation when there are two squares left.

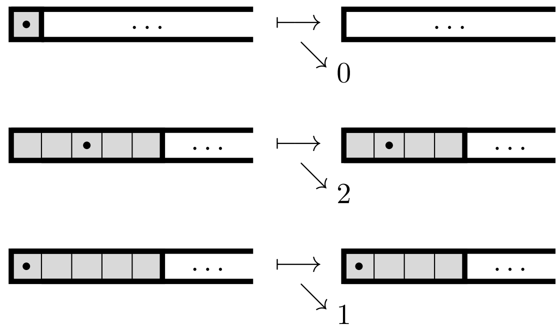

Figure 7. Generating a ternary string from an n-color composition.

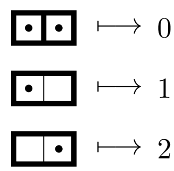

Figure 8. Three possibilities of the end of the process.

2.3. **Between the binary words and the ternary strings.** It is not hard to notice the similarity between the two aforementioned bijections. Indeed, there is a natural bijection between the binary words with exactly n strictly increasing runs (where $|0|1|$ is "avoided") and the $\{0, 1, 2\}$ strings of length n that avoids 12 through the following:

- $0 \rightleftharpoons |01|$;
- $1 \rightleftharpoons |0|$;
- $2 \rightleftharpoons |1|$.

3. Concluding Remarks and Potential Applications

In this note we make use of the spotted tiling representation of an n-color composition to show bijections between such compositions and other objects counted by the bisection of Fibonacci sequence. The bijection to binary words with $n-1$ strictly increasing runs seems to be particularly interesting.

This bijection offers a generalization of the classical bijection between compositions of n and binary words of length $n-1$. This classical bijection has been useful in many bijective arguments between compositions. See [11] for a recent example of such applications.

Much information regarding the n-color composition can be readily obtained from the corresponding binary word. For instance, the number of parts of the composition is one plus the number of runs $|01|$. One can compare this observation with the classical case (1.1) where the number of parts is one plus the number of 0's. Similarly, the number of parts of an n-color composition is one more than the number of 0's in the corresponding ternary string without 12.

By taking the "conjugate" of a binary word (i.e., exchange 1 and 0), the classical result yields a number of interesting one-to-one correspondences between compositions with various constraints (for example, [11]). In terms of the n-color compositions and binary words with increasing runs, it is the number of increasing runs (corresponding to the number whose composition we are considering) instead of the number of digits in a binary word that is of importance. It is not difficult to observe the following.

Proposition 3.1. *The conjugate of a binary word that starts and ends with the same digit has the same number of increasing runs as that of the original binary word.*

With Proposition 3.1, we can define conjugates for n-color compositions whose corresponding binary words start and end with the same digit. For example, the n-color composition

$$3_3 + 4_1 + 2_2 = 9$$

is mapped to the binary word

$$1|1|01|0|0|0|01|1$$

that starts and ends with the same digit 1. Taking the conjugate yields

$$0|01|01|1|1|1|0|0$$

that has 8 increasing runs, the same as the original binary word. Hence we have the n-color composition

$$2_1 + 1_1 + 6_4 = 9$$

as the conjugate.

The above discussion does not apply to binary words that start and end with different digits. In fact, the following is easily observed.

Proposition 3.2. *For a binary word that starts with 1 and ends with 0, the conjugate will have one fewer increasing run. Similarly, for a binary word that starts with 0 and ends with 1, the conjugate will have one more increasing run.*

For example, consider the n-color composition

$$3_2 + 4_4 + 1_1 + 5_3 = 13$$

from Figure 5, the corresponding binary word is

$$1|0|01|1|1|1|01|01|1|1|0|0,$$

a binary word with 12 increasing runs starting with 1 and ending with 0. Now taking the conjugate yields

$$01|1|0|0|0|01|01|0|0|01|1,$$

a binary word with $12 - 1 = 11$ increasing runs. The corresponding n-color composition is then obtained as

$$1_1 + 5_2 + 1_1 + 3_1 + 2_2 = 12.$$

From Proposition 3.2, one can still define the "conjugate" of such an n-color composition similarly. The "conjugate" defined, however, will map compositions of a number n to $n+1$ or $n-1$ (the "conjugate" of the "conjugate" will return to the original composition). As a referee suggested, one might be able to take advantage of the ternary strings without 12 (for which the compositions will be of the same number, as are the lengths of the strings) provided that a novel "conjugation" of them can be devised.

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