INDEPENDENT SETS OF CARDINALITY s OF MAXIMAL OUTERPLANAR GRAPHS

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ABSTRACT. In 1982, Prodinger and Tichy defined the Fibonacci number $f(G)$ to be the number of independent sets in the graph G. Let $\alpha(G)$ be the cardinality of a maximum independent set of G and $f_s = f_s(G)$ be the number of independent sets of cardinality s in G. Then the independence polynomial of G is defined to be $I(G;x) = \sum_{s=0}^{\alpha(G)} f_s(G)x^s$, and so $I(G;1) = f(G)$. In 1998, Alameddine determined that $f(P_n^2) \leq f(G) \leq f(S_n^2)$ for maximal outerplanar graphs G with equality reached uniquely by the 2-path P_n^2 and the 2-spiral S_n^2, respectively. We will investigate $f(G)$ for maximal outerplanar graphs by way of the coefficients f_s of $I(G;x)$; we show that for a maximal outerplanar graph G and $s \geq 3$, $\binom{n+2-2s}{s} \leq f_s(G) \leq \binom{n-s}{s}$. The lower bound is uniquely reached by P_n^2, and the upper bound is reached exclusively by D_6^2 and S_n^2. As a corollary, we show the works of Alameddine with one more graph obtaining the upper bound when $n = 6$.

Throughout this paper $G = (V, E)$ is a finite simple undirected graph. For graphs G and H, $G \cong H$ denotes that G is isomorphic to H. Let $G \vee H$ denote the graph obtained by adding an edge from each vertex in G to each vertex in H. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and $G - S$ denotes $G[V(G) - S]$. For a vertex $v \in V(G)$, we let $N(v) = \{u | u \in V(G), uv \in E(G)\}$, $N[v] = N(v) \bigcup \{v\}$, and the degree of v, $d(v) = |N(v)|$. Let $\delta = \min\{d(v) | v \in V(G)\}$. We use K_n, P_n, and $S_{1,n-1}$ for a clique, a path, and a star, all on n vertices, respectively.

An independent set in a graph G is a set of pairwise non-adjacent vertices. Let $\alpha(G)$ denote the cardinality of a maximum independent set of G. In 1982, Prodinger and Tichy [7] defined the Fibonacci number $f(G)$ to be the total number of independent sets of G. Let $f_s = f_s(G)$ be the number of independent sets of cardinality s of G. Then the polynomial

$$I(G;x) = \sum_{s\geq 0}^{\alpha(G)} f_s(G)x^s$$

is called the independence polynomial [3], the independent set polynomial [4], or Fibonacci polynomial [5] of G. It is clear then that $I(G;1) = f(G)$. Fibonacci numbers and independence polynomials have been areas of vigorous research, and for results not presented here we refer the reader to a thorough survey paper by Levit and Mandrescu [6].

The following proposition is commonly known and will be used in subsequent sections.

Proposition 1. *Let G be a graph on n vertices and m edges, and let $v \in V(G)$. Then*

(i) $f_0(G) = 1$
(ii) $f_1(G) = n$
(iii) $f_2(G) = \binom{n}{2} - m$
(iv) $f_s(G) = f_s(G - v) + f_{s-1}(G - N[v])$ *for* $s \geq 1$.
(v) *Let H be a spanning subgraph of G. Then* $f_s(G) \leq f_s(H)$.

An outerplanar graph is a graph that has a planar drawing with all vertices on the same face. A graph G is maximal outerplanar if G is outerplanar and is not a spanning subgraph of another outerplanar graph. We say an edge of a maximal outerplanar graph G is bound if it is contained in two triangles of G and unbound if it is not bound. Then it is easy to verify that a maximal outerplanar graph G has a unique Hamiltonian cycle that passes through the unbound edges of G. Also for $v \in V(G)$, if $d(v) = 2$, then $G[N(v)] \cong K_2$ and $G - v$ is also a maximal outerplanar graph. Define the 2-path, P_n^2, to be the maximal outerplanar graph with vertex set $\{v_1, \ldots, v_n\}$ where $G[\{v_1, v_2\}] \cong K_2$, and for $3 \leq i \leq n$, vertex v_i is adjacent to vertices $\{v_{i-1}, v_{i-2}\}$. Define 2-spiral as $S_n^2 \cong K_1 \vee P_{n-1}$, and define D_6^2 to be the unique maximal outerplanar graph on six vertices with three vertices of degree 2.

In 1998, Alameddine determined sharp bounds of the Fibonacci number of maximal outerplanar graphs and characterized the unique maximal outerplanar graphs that obtained these bounds. He found the following.

Theorem 2. [1] *Let G be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \geq f(P_n^2)$, and equality is reached if and only if $G \cong P_n^2$.*

Theorem 3. [1] *Let G be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \leq f(S_n^2)$, and equality is reached if and only if $G \cong S_n^2$.*

We note for $n = 6$, $f(S_6^2) = f(D_6^2) = 14$, and thus Theorem 3 is not complete. In this paper, we will demonstrate a revision of the results of Alameddine including this special case through investigating lower and upper bounds of the coefficients of $I(G;x)$, $f_s(G)$ for $s \geq 0$. Additionally, we will classify the unique graphs that obtain these bounds.

Now a maximal outerplanar graph G on n vertices has $2n+3$ edges. Hence by Proposition 1, it is clear that $f_s(G_1) = f_s(G_2)$ for $0 \leq s \leq 2$ for maximal outerplanar graphs G_1 and G_2. Thus we need to only consider $s \geq 3$. Also, as there is only one maximal outerplanar graph on $n \in \{3, 4, 5\}$ vertices, we need to only consider $n \geq 6$. We must first introduce some related concepts.

In 1968, Beineke and Pippet introduced the notion of k-trees [2], which will now be defined. For $n = k$, $G \cong K_k$. Let G be a k-tree on $n \geq k+1$ vertices. Add a new vertex v such that $G[N(v)] \cong K_k$. The resulting graph is a k-tree on $n+1$ vertices. By this definition, it is clear that maximal outerplanar graphs are 2-trees.

In 2010, Song, Staton, and Wei determined sharp lower bounds for f_s of k-trees for $s \geq 3$ and determined the unique graphs that obtained these bounds [8]. For $k = 2$, the 2-tree that uniquely obtains this sharp lower bound is the 2-path. Thus the lower bound of f_s for maximal outerplanar graphs for $s \geq 3$ immediately follows by the results of Song et. al. with $k = 2$. We may state the following theorem.

Theorem 4. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices. Then for all $s \geq 3$,*

$$f_s(G) \geq \binom{n+2-2s}{s},$$

and equality holds if and only if $G \cong P_n^2$.

Corollary 5. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices. Then $f(G) \geq f(P_n^2)$ with equality holding if and only if $G \cong P_n^2$.*

We note that Corollary 5 is consistent with the results of Alameddine.

We will now focus on determining the sharp upper bound.

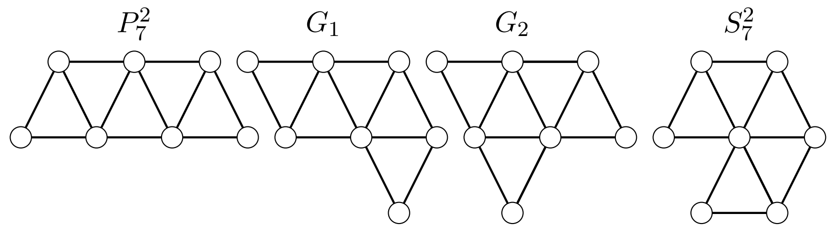

FIGURE 1. Maximal outerplanar graphs on $n = 7$ vertices.

Theorem 6. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices. Then for all $s \geq 3$,*

$$f_s(G) \leq \binom{n-s}{s},$$

and equality holds if and only if $G \in \{S_6^2, D_6^2\}$ for $n = 6$ and $G \cong S_n^2$ for $n \geq 7$.

Proof. First, we note that in 1984, Hopkins and Staton determined that, for the path P_n on n vertices and $s \geq 0$, $f_s(P_n) = \binom{n+1-s}{s}$ [5] which will be used in our proofs. We need only to consider $s \geq 3$. Suppose $n = 6$. Then $G \in \{P_6^2, S_6^2, D_6^2\}$. Let $n = 7$. Then $G \in \{P_7^2, G_1, G_2, S_7^2\}$ as pictured in Figure 1. Routine calculations show that for $n \in \{6, 7\}$, $\alpha(G) \leq 3$, $f_3(P_6^2) = 0$, $f_3(D_6^2) = f_3(S_6^2) = 1 = \binom{6-3}{3}$, $f_3(P_7^2) = 1$, $f_3(G_1) = 2$, $f_3(G_2) = 3$, and $f_3(S_7^2) = 4 = \binom{7-3}{3}$. Thus the theorem holds for $n \in \{6, 7\}$.

Suppose that for maximal outerplanar graphs on $7 \leq n' < n$ vertices the theorem holds, and let G be a maximal outerplanar graph on $n \geq 8$ vertices. Let $v \in V(G)$ such that $d(v) = 2$ and $N(v) = \{u_1, u_2\}$. By Proposition 1(iv)

$$f_s(G) = f_s(G - v) + f_{s-1}(G - N[v]), \tag{1}$$

and as $G - v$ is a maximal outerplanar graph by induction, $f_s(G - v) \leq f_s(S_{n-1}^2) = \binom{n-1-s}{s}$.

Now G has a Hamiltonian cycle C that passes through all of the unbound edges of G. Thus $u_1 v u_2$ is a segment of C, and so $G - N[v]$ has a spanning path on $n - 3$ vertices, namely $C - N[v]$. By Proposition 1, $f_{s-1}(G - N[v]) \leq f_{s-1}(P_{n-3}) = \binom{n-3+1-(s-1)}{s-1} = \binom{n-1-s}{s-1}$.

Thus by induction and (1),

$$\begin{aligned}
f_s(G) &= f_s(G - v) + f_{s-1}(G - N[v]) \\
&\leq f_s(S_{n-1}^2) + f_{s-1}(P_{n-3}) \\
&= \binom{n-1-s}{s} + \binom{n-1-s}{s-1} \\
&= \binom{n-s}{s-1},
\end{aligned}$$

and for $s \geq 3$ equality holds if and only if $G - v \cong S_{n-1}^2$ and $G - N[v] \cong P_{n-3}$, i.e. $G \cong S_n^2$. $\square$

As a corollary, we obtain the following modified result of Alameddine.

Corollary 7. *Let G be a maximal outerplanar graph on $n \geq 6$. Then $f(G) \leq f(S_n^2)$. If $n = 6$, then equality is reached if and only if $G \in \{D_6^2, S_6^2\}$, and if $n \geq 7$, $G \cong S_n^2$.*

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