COUNTING REARRANGEMENTS ON GENERALIZED WHEEL GRAPHS

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Abstract. In this paper we pose and answer a variety of combinatorial questions concerning cycle covers and matchings on graph structures. Particularly, we study wheel graphs and their natural extensions. These problems are motivated by a generalization of a seating rearrangement model explored by Kennedy and Cooper in the 1990's. We provide answers in terms of closed form expressions and demonstrate some interesting relationships between these counting problems and the Fibonacci numbers.

1. Introduction

1.1. **Background.** Wheel graphs are a simple and familiar combinatoric structure that give rise to numerous counting problems. The wheel graph of order n, denoted W_n, is defined as an n–cycle with one additional vertex that is adjacent to each of the vertices in the cycle. Thus, W_n has $n+1$ vertices and $2n$ edges (Figure 1). Many counting problems on wheel graphs have already been considered and can be found in the literature. These problems include enumerating the number of cycles on a wheel graph, counting the number of matchings on a wheel graph, and computing the number of spanning trees on a wheel graph. Indeed, the OEIS has over 20 different entries related to counting problems defined on wheel graphs and the sequences that these problems generate [10].

In this paper we describe some natural variations of the basic wheel graph and consider a variety of traditional counting problems applied to these structures. The solutions to these problems are given as closed form representations that can also be expressed as recurrence relations. Most of our solutions also include Fibonacci numbers and other well-known combinatorial entities. We begin by defining the new structures that we are considering, and by providing a combinatorial interpretation of the perfect matchings and cycle covers that we are counting.

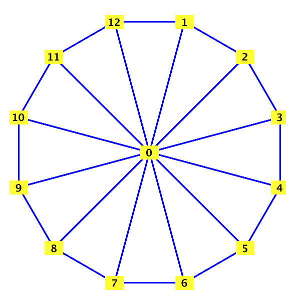

Figure 1. W_{12}, with 13 vertices and 24 edges.

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1.2. **Motivation.** In the 1990's, Kennedy and Cooper proposed a set of counting problems based on seating rearrangements [6], motivated by a problem presented by Honsberger [5]. They considered a classroom with desks arranged in an $m \times n$ rectangle, and counted the number of seating rearrangements that could be performed assuming that there was originally a student in each desk. To construct a permissible rearrangement, the students are each required to move to an adjacent desk, while remaining in either their original column or row. In addition, a legitimate rearrangement required that every desk be filled by exactly one student. In their second paper, Otake, Kennedy, and Cooper provide a closed form solution to this problem for the general $m \times n$ case [11].

This set of problems can be easily restated in graph–theoretic terminology. A seating rearrangement of an $m \times n$ classroom can be described as a cycle cover on a $m \times n$ grid graph, where the graph has been modified by replacing each of its edges with two directed edges, one in each orientation. This generalization makes it natural to consider rearrangement problems on non–grid graphs using the following problem statement.

Problem. *Given a graph, place a marker on each vertex. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:*

- *Each marker must move to an adjacent vertex.*
- *After all of the markers have moved, each vertex must contain exactly one marker.*

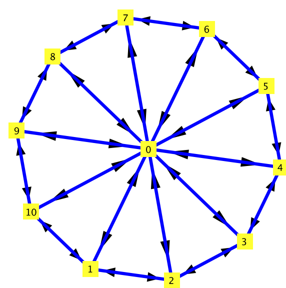
It is also natural to consider counting rearrangements where the markers are permitted to either move to an adjacent vertex or remain in their original position. This modification can be performed by adding a self-loop to each vertex in the graph, forming a pseudograph. Then, the number of cycle covers on the pseudograph is equal to the number of rearrangements, where each marker either moves to an adjacent vertex, or remains in place. Applying this procedure to wheel graphs leads to the results contained in this paper.

Counting the number of cycle covers on a digraph is equivalent to computing the permanent of the adjacency matrix of the digraph [4]. This permanent also counts the number of perfect matchings in a bipartite graph [4]. However, computing the permanent of a general matrix is $\#P$–complete [12], although there do exist polynomial–time algorithms for the adjacency matrices of planar graphs, by reduction to a pfaffian using Kasteleyn's method [7]. Computations of perfect matchings on combinatorial objects with permanents and determinants, as well as pfaffians, are discussed in [7, 8, 9].

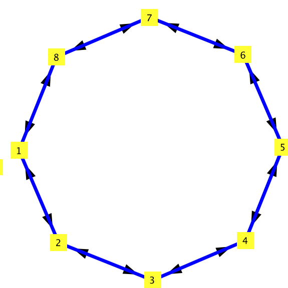
1.3. **Definitions.** This section contains a description of the structures that we are counting and the notation used to describe them. Throughout this paper, any reference to the Fibonacci numbers f_n will be understood to mean the combinatorial Fibonacci numbers. As was stated above, W_n will represent the wheel graph of order n. In addition, P_n and C_n will respectively represent the traditional path and cycle graphs on n vertices. In a wheel graph, the vertex that is adjacent to each other vertex will be called the center of the graph, while the edges incident to this vertex will be called the spokes of the graph. The non–center vertices that lie along the original cycle comprise the rim of the graph.

Motivated by the seating rearrangement problem, in this paper we pose counting problems not only for traditional, simple graphs, but also on variations of these graphs called pseudographs. To form a pseudo-graph from a given graph, we will simply add a self-loop to each vertex. These graphs will be denoted with a superscript p. In addition, we will form directed graphs from ordinary graphs by replacing each undirected edge between two separate vertices with two directed edges, one with each orientation. These directed graphs will be denoted by placing a double-sided arrow, $\leftrightarrow$, over the symbol for the respective original graph or pseudograph. Thus, $\overleftrightarrow{W_8^p}$ is the directed pseudo-wheel with 9 vertices, while $\overleftrightarrow{C_8}$ is the directed 8–cycle (Figure 2).

We will be interested in counting the number of perfect matchings and cycle covers for each of the graph structures that we have defined. A perfect matching on a given graph, $G = (V, E)$, is a subset of the edges, $M \subseteq E$, such that the induced subgraph on M contains each vertex in V, and each vertex in the induced subgraph has degree 1 [3]. A cycle cover, or linear subgraph, on a graph is a subset of the edges, $C \subseteq E$, such that the induced subgraph of C contains each vertex in V, and each vertex in the induced subgraph lies on exactly one cycle [4]. Thus, in a cycle cover on an undirected graph each vertex has degree 2, while in a directed graph each vertex has in–degree 1 and out–degree 1.



(A) $\overleftrightarrow{W_{10}}$



(B) $\overleftrightarrow{C_8}$

Figure 2. Directed Graphs.

2. Counting Problems

2.1. **Simple Graphs.** In this section we pose and answer a variety of counting problems concerning the structures defined previously. For the sake of completeness, we begin by presenting some basic results on the number of matchings and cycle covers of the simple wheel graph of order n.

Proposition 2.1. *The number of perfect matchings of W_n is equal to 0 if n is even or n if n is odd.*

Proof. When n is even there are an odd number of vertices in W_n and hence there can be no perfect matchings in W_n. When n is odd, there are exactly n vertices on the cycle to pair with the center. This choice uniquely defines the matching and thus there are n total perfect matchings. □

Proposition 2.2. *The number of cycle covers of W_n is equal to n.*

Proof. It is easy to see that any cycle cover of W_n must consist of a single cycle that contains each vertex of W_n. This implies that in a cycle cover, the center vertex must be adjacent to two consecutive vertices along the original n–cycle. Since there are n such pairs of vertices, there are exactly n cycle covers on W_n. □

We next consider the number of cycle covers that occur on $\overleftrightarrow{W_n}$. The following proof is interesting because a separate argument is needed for each parity, but the same expression counts both cases.

Theorem 2.3. *The number of cycle covers of* $\overleftrightarrow{W_n}$ *is equal to* n^2.

Proof. Any cycle cover of $\overleftrightarrow{W_n}$ must consist of a cycle containing the center and at least one other vertex. The remaining vertices (if any) must pair up in a series of 2–cycles. Thus, the number of vertices that do not lie on a cycle with the center vertex must be even. This suggests we divide our proof into two cases based on the parity of n. Regardless of parity, there are n ways to choose a vertex, call it v, such that there is an edge oriented from the center to v in the cycle cover.

Now, assume that n is odd. We must select another vertex, u, not necessarily distinct from v, along the rim, to have an edge in the cycle cover oriented towards the center, and an orientation from v to u. Since n is odd, any choice of v and u partitions the rim into two subsets, one with even parity and one with odd parity. Since there must be an even number of vertices that do not lie on the cycle that contains the center vertex, and thus on the path from u to v, there is only one choice of orientation of the (u, v) path. Thus, each choice of v and u defines a unique cycle cover and there are $n \cdot n = n^2$ cycle covers if n is odd.

Finally, assume that n is even. We must again select a vertex u on the rim that allows us to create a legitimate cycle cover. Selecting any vertex u along the rim we see that if the distance from v to u is odd then the rim vertices are partitioned into two sets of odd cardinality while if the distance between v and u is even the rim vertices are partitioned into two sets with even cardinality. Thus, in order to construct a legitimate cycle cover, there must be an even number of vertices that separate vertex u from vertex v along the rim. There are exactly $\frac{n}{2}$ vertices that satisfy this condition for any of the n choices for v. However, since v and u partition the cycle into two even subsets, there are two possible orientations for every permissible choice of v and u. Thus, there are $n\left(\frac{n}{2} \cdot 2\right) = n^2$ cycle covers if n is even, which completes the proof. □

Before stating and proving the remaining propositions we need some preliminary results that define the number of perfect matchings and cycle covers on some simpler graph structures. Then we can use these components to count our final problems more easily.

Lemma 2.4. *The number of cycle covers of* $\overleftrightarrow{P_n^p}$ *is equal to the number of perfect matchings of* P_n^p*, and both of these values are equal to* f_n.

Proof. This follows directly from the interpretation of the combinatorial Fibonacci numbers as the number of ways to tile a $1 \times n$ rectangle with squares and dominoes. We can provide a bijection between these problems by associating a vertex whose self-edge occurs in the matching or cycle cover with a square in the tiling and associating each pair of vertices connected by an edge in the matching or a 2–cycle in the cycle cover to a domino in the tiling. □

Lemma 2.5. *The number of perfect matchings of* C_n^p *is equal to* $f_n + f_{n-2}$.

Proof. Using the bijection that was presented in the previous lemma, we see that this problem is equivalent to "tiling" the graph with "squares" and "dominoes". Labeling the vertices from 1 to n we can condition on whether or not the edge $(1, n)$ is in the matching. If that edge is not in the matching then we are left with counting the number of perfect matchings in P_n^p which we showed in the previous lemma to be f_n. If that edge is included in the matching then we are counting the number of perfect matchings in P_{n-2}^p of which there are f_{n-2}. Since these cases span all possibilities, the number of perfect matchings is their sum $f_n + f_{n-2}$ and this proof is complete. □

A result equivalent to Lemma 2.5 is presented by Benjamin and Quinn in their book [2]. They present this result as a combinatorial interpretation of the Lucas numbers, l_n with

$l_0 = 2$, $l_1 = 1$ and $l_n = l_{n-1} + l_{n-2}$ for $n \geq 2$. Using the well–known Fibonacci–Lucas identity $l_n = f_n + f_{n-2}$ it is easy to see the relationship between the two motivations [1].

Lemma 2.6. *The number of cycle covers of* $\overleftrightarrow{C_n^p}$ *is equal to* $f_n + f_{n-2} + 2$.

Proof. All of the perfect matchings counted in the previous lemma directly correspond to cycle covers in $\overleftrightarrow{C_n^p}$ by simply replacing any edge in a matching between two different vertices by a 2–cycle. However, in $\overleftrightarrow{C_n^p}$ there are also the two additional directed cycles that contain each vertex in the graph. This gives the plus 2 term in the total and this proof is complete. □

2.2. **Wheel Graphs.**

Theorem 2.7. *The number of perfect matchings of* W_n^p *is equal to* $f_n + nf_{n-1} + f_{n-2}$.

Proof. We can condition on the behavior of the center vertex. If the center vertex is matched to itself, then we are left counting the number of perfect matchings in C_n^p which we showed in Lemma 2.5 to be equal to $f_n + f_{n-2}$. If the center is matched to a vertex on the cycle the remaining vertices from P_{n-1}^p and we showed in Lemma 2.4 that there are f_{n-1} perfect matchings on this structure. Since we can match the center vertex to any one of the n other vertices, summing over these cases gives $f_n + nf_{n-1} + f_{n-2}$ perfect matchings. □

Theorem 2.8. *The number of cycle covers of* W_n^p *is equal to* $n(n-1) + 2$.

Proof. We can again condition on the behavior of the center vertex. If the center vertex lies on a 1–cycle then we are counting the number of cycle covers in C_n^p of which there are two: one where each vertex on the rim is connected, and one where each vertex on the rim lies on a 1–cycle. If the center vertex lies on a cycle with $k \geq 2$ other vertices, then the cycle cover will consist of that cycle and $n - k$ additional 1–cycles. Thus, each choice of two vertices to be adjacent to the center, in the cycle cover, determines two unique cycle covers because there are two available paths along the rim for any choice of two such vertices. Since there are $\binom{n}{2} = \dfrac{n(n-1)}{2}$ ways to choose the two vertices, there are $2\binom{n}{2} = n(n-1)$ cycle covers where the center vertex is adjacent to at least two other vertices. Summing these two cases gives the desired result. □

We conclude this section with a most interesting result, uncovering another relationship to the Fibonacci numbers, by counting the number of rearrangements on a wheel graph when the markers are permitted either to move to an adjacent vertex or remain in place.

Theorem 2.9. *The number of cycle covers of* $\overleftrightarrow{W_n^p}$ *is equal to* $nf_{n+2} + f_n + f_{n-2} - 2n + 2$.

Proof. We again condition on the behavior of the center vertex. If the center vertex lies on a 1–cycle then we are counting the number of cycle covers on $\overleftrightarrow{C_n^p}$ of which there are $f_n + f_{n-2} + 2$ as was shown in Lemma 2.6.

There are n ways the center vertex can lie on a 2–cycle. For each of these the remaining vertices form $\overleftrightarrow{P_{n-1}^p}$ on which there are f_{n-1} cycle covers.

If the center vertex lies on a cycle that contains $k \geq 2$ other vertices, the vertices not on the cycle form $\overleftrightarrow{P_{n-k}^p}$, on which there are f_{n-k} cycle covers by Lemma 2.4. To form a $k+1$ cycle containing the center vertex, we select in n ways the vertex adjacent to the center and either a clockwise or counterclockwise orientation. Thus for each value of $2 \leq k \leq n$ there are

$2n$ distinct $(k+1)$–cycles that contain the center, each of which determines f_{n-k} unique cycle covers.

Thus, there are $2n\sum_{k=2}^{n} f_{n-k}$ cycle covers where the center lies on a cycle of length 3 or greater. It is a well-known Fibonacci identity that $\sum_{i=0}^{n} f_i = f_{n+2} - 1$ [1]. Applying this identity to our sum gives the following

$$2n\sum_{k=2}^{n} f_{n-k} = 2n\,(f_n - 1) = 2nf_n - 2n.$$

We can now combine our three cases to obtain $f_n + f_{n-2} + 2 + nf_{n-1} + 2nf_n - 2n$. Using the definition of the Fibonacci numbers, we can simplify our expression as follows:

$$\begin{aligned} f_n + f_{n-2} + 2 + nf_{n-1} + 2nf_n - 2n &= n((f_{n-1} + f_n) + f_n) + f_n + f_{n-2} - 2n + 2 \\ &= n(f_{n+1} + f_n) + f_n + f_{n-2} - 2n + 2 \\ &= nf_{n+2} + f_n + f_{n-2} - 2n + 2. \end{aligned}$$

Thus, there are $nf_{n+2}+f_n+f_{n-2}-2n+2$ cycle covers on $\overleftrightarrow{W_n^p}$ and this proof is complete. □

3. Modified Wheel Graphs

In this section we define two modified types of wheel graphs, flat–wheel graphs and $k-$wheel graphs and consider counting problems, similar to those posed above, on these new structures.

As defined previously, the wheel graph of order n can be thought of as an $n-$cycle with an additional vertex adjacent to each of the vertices in the cycle. We will define the flat–wheel graph of order n, denoted FW_n, as a similar structure, where the cycle of order n is replaced with a path on n vertices. Figure 3 displays a flat–wheel graph. We will adopt the convention that the vertices in the path are labeled from left to right, with the integers $1, 2, \ldots, n$. Also, we will refer to the vertex that dominates each of the path vertices as the center to maintain consistency.

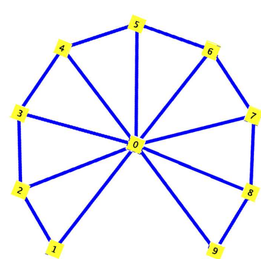
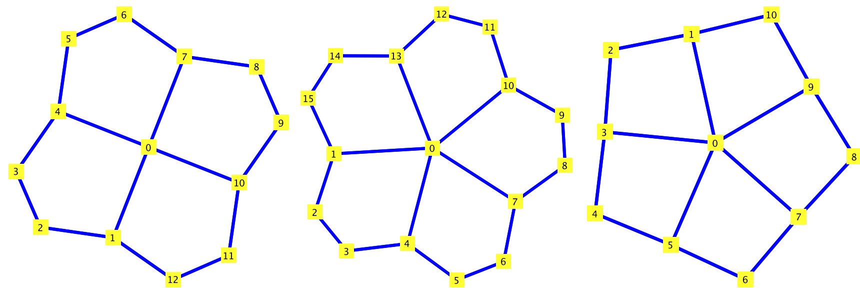


Figure 3. The Flat–Wheel of Order 9.

Additionally, we define the $k-$wheel graph of order n, denoted ${}_kW_n$, as a different generalization of the traditional wheel graph. A standard wheel graph can be considered as a collection of n triangles, each of which share a common vertex, where each triangle further shares an edge with each of its neighbors. We wish to extend this notion to larger cycles. Thus, ${}_kW_n$ represents n copies of a $k-$cycle that share a common vertex, and where each $k-$cycle shares an edge with its neighboring cycles. Under this definition then, ${}_3W_n$ is the standard wheel graph of order n, while ${}_5W_n$ represents a collection of 5$-$cycles joined at a vertex and sharing a single edge between each pair of neighboring cycles (Figure 4).



(A) ${}_5W_4$ (B) ${}_5W_5$ (C) ${}_4W_5$

FIGURE 4. $k-$Wheel Graph Examples.

The next two counting problems employ arguments very similar to the one introduced in Theorem 2.3.

Theorem 3.1. *The number of cycle covers on* $\overleftrightarrow{FW_n}$ *is equal to* $\dfrac{n^2+2n+1}{4}$ *when* n *is odd and* $\dfrac{n^2+2n}{4}$ *when* n *is even.*

Proof. Any cycle cover on $\overleftrightarrow{FW_n}$ must consist of a single cycle of length $k \geq 2$ that contains the center vertex and a pairing of the remaining $n+1-k$ vertices into $2-$cycles. Thus, $n+1-k$ must be even, and the vertices not lying on a cycle with the center vertex must be adjacent in pairs.

First, assume that n is odd. Then, as noted above the center vertex must lie on a cycle of even length. In order to construct a cycle of even length that contains the center vertex and allows the vertices not on the cycle to form adjacent pairs, the vertices adjacent to the center in the cycle cover must have odd labels. Since n is odd, there are $\frac{n+1}{2}$ odd labeled vertices. We can independently select two of these vertices to be adjacent to the center in the cycle cover, which uniquely defines the cycle cover. Hence, there are $\frac{n+1}{2}\frac{n+1}{2} = \frac{n^2+2n+1}{4}$ cycle covers when n is odd and this case is complete.

Now assume that n is even. Then, the cycle that contains the center vertex must be of odd length. Note that there are two possible $n+1-$cycles that contain every vertex. Similarly, there are four legitimate $n-1-$cycles and more generally, $n-m+3$ cycles of length m where m is an odd integer, $3 \leq m \leq n+1$. Thus, the number of cycle covers is equal to

$$\sum_{i=1}^{\frac{n}{2}} 2i = \frac{n^2+2n}{4}.$$

$\square$

Theorem 3.2. *The number of cycle covers on* $\overleftrightarrow{{}_kW_n}$ *is equal to* 0 *when both* k *and* n *are even and* n^2*, otherwise.*

Proof. First note that when n and k are both even, the graph is bipartite and has an odd number of vertices, thus by the pigeonhole principle no cycle covers can exist. In the case when at least one of k or n is odd we may follow the proof of Theorem 2.3 directly. Note that Theorem 2.3 is a special case of this Theorem, with $k=3$.

When n is odd, any selection of the vertices to be adjacent to the center determines a unique cycle cover since this choice again partitions the rim vertices into two sets of opposite parity. When n is even, any choice of vertex to receive a directed edge from the center in the cycle cover leaves a choice of $\frac{n}{2}$ vertices to have an edge oriented towards the center in the cycle cover. There are two legitimate paths between any two vertices adjacent to the center in a cycle cover, and hence this proof is complete. □

Theorem 3.3. *The number of cycle covers on* $\overleftrightarrow{FW_n^p}$ *is equal to*

$$f_n + \sum_{\ell=1}^{n} [f_{n-\ell}(f_{\ell+1} - 1) + f_{\ell-1}(f_{n-\ell+1} - 1)].$$

Proof. We will construct our argument based on the behavior of the center vertex. In the simplest case, the center vertex lies on a 1−cycle. Then, the remaining vertices form $\overleftrightarrow{P^p}$, on which there are f_n cycle covers. Recalling that the path vertices are labeled from $1, \ldots, n$ from left to right, when the center vertex does not lie on a 1−cycle, it must have a directed edge towards one of the n path vertices with label ℓ.

The cycle that contains the center vertex and ℓ must fall into one of three types:

(1) A 2−cycle containing only the center and ℓ.
(2) A cycle of length $k > 2$ that contains vertex $\ell - 1$.
(3) A cycle of length $k > 2$ that contains vertex $\ell + 1$.

Note that vertex 1 cannot have any rearrangements of type 2, while vertex n cannot have any rearrangements of type 3.

In case 1 the remaining vertices are partitioned into at most two sets, P_{n-l} and $P_{\ell-1}$. Thus, there are $f_{n-\ell}f_{\ell-1}$ rearrangements of this form for each $1 \leq \ell \leq n$.

In case 2 there are $\ell - 1$ cycles of lengths $3 \leq m \leq \ell + 1$, each of which divides the remaining vertices into P_{n-l} and $P_{\ell-m+1}$. Thus there are

$$f_{n-\ell} \sum_{j=0}^{\ell-2} f_j = f_{n-\ell}(f_\ell - 1)$$

cycle covers for each choice of ℓ of type 2.

Finally, in case 3 there are $n - \ell$ cycles of lengths $3 \leq m \leq n - \ell + 2$, each of which divides the remaining vertices into $P_{\ell-1}$ and $P_{n-\ell-m+2}$. Thus, there are

$$f_{\ell-1} \sum_{k=0}^{n-\ell-1} f_k = f_{\ell-1}(f_{n-\ell+1} - 1)$$

cycle covers for each choice of ℓ of type 3.

Summing over all possible values of ℓ gives the following:

$$f_n + \sum_{\ell=1}^{n} [(f_{n-\ell}(f_\ell - 1)) + (f_{\ell-1}f_{n-\ell}) + (f_{\ell-1}(f_{n-\ell+1} - 1))],$$

which can be simplified by factoring and the Fibonacci recurrence to give the result. □

We conclude with a theorem that generalizes the result of Theorem 2.9 to k−wheel graphs.

Theorem 3.4. *The number of cycle covers on* ${}_k\overleftrightarrow{W_n}^p$ *is equal to*

$$l_{(k-2)n} + 2 + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} + 2n\sum_{i=1}^{k-2} f_{(k-2)(n-i-1)-1}.$$

Proof. We may again condition on the behavior of the center vertex. When the center lies on a 1−cycle, there remaining vertices form $\overleftrightarrow{C^p_{(k-2)n}}$ which we showed in Lemma 2.6 has $l_{(k-2)n} + 2$ cycle covers. Furthermore, there are n ways in which the center vertex lies on a 2−cycle. In each of these cases the remaining vertices form $P_{(k-2)n-1}$ which has $f_{(k-2)n-1}$ cycle covers by Lemma 2.4.

The center may lie on one of $2n$ legitimate n−cycles each of which leaves a $P_{(n-2)k-(n-1)}$ with $f_{(n-2)k-(n-1)}$ cycle covers. When the center vertex lies on a longer cycle, that cycle must have length $l = k + (k-2)i$, where i is a positive integer less than $k-1$. Selecting one of the n vertices adjacent to the center, and an orientation for the cycle, gives $2n(k-2)$ legitimate cycles. Each of these cycles leaves a path of length $(k-2)n-(k-2)(i+1)-1 = (k-2)(n-i-1)-1$. Applying Lemma 2.4 a final time and summing over all possible values of i gives the final result. □

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