ON THE COUNTING FUNCTION OF TRIPLES WHOSE PAIRWISE PRODUCTS ARE CLOSE TO FIBONACCI NUMBERS

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ABSTRACT. For a positive real number x let the Fibonacci distance $\|x\|_F$ be the distance from x to the closest Fibonacci number. We let

$$f(x) = \#\{(a,b,c) \in \mathbb{Z}^3 : a > b > c \geq 1,\ \max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \leq x\}$$

and study the function $f(x)$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. For a positive real number x we let

$$\|x\|_F = \min\{|x - F_n| : n \geq 0\}. \tag{1.1}$$

In [1], it was shown that if $a > b > c \geq 1$ are integers then

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log a}). \tag{1.2}$$

Here, we revisit the Fibonacci distances of ab, ac, and bc for positive integers a, b, and c. We define the function

$$f(x) = \#\{(a,b,c) \in \mathbb{Z}^3 : a > b > c \geq 1,\ \max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \leq x\}. \tag{1.3}$$

We study the behavior of $f(x)$ as $x \to \infty$. We have the following result.

Theorem 1.1. *The estimates*

$$x^{3/2} \ll f(x) \leq x^{2+o(1)}$$

hold as $x \to \infty$.

For the non-negative integers $x \leq 2$ we obtain the following theorem.

Theorem 1.2.

$$f(0) = 0, \qquad f(1) = 16, \qquad f(2) = 49.$$

Throughout the paper, we use the Landau symbols O and o as well as the Vinogradov symbols $\ll$, $\gg$, and $\asymp$ with their regular meanings. Recall that $F = O(G)$, $F \ll G$ and $G \gg F$ are all equivalent and mean that the inequality $|F| \leq cG$ holds with some constant c, whereas $F \asymp G$ means that both inequalities $F \ll G$ and $G \ll F$ hold. The constants implied by these symbols are absolute. Further, $F = o(G)$ means that $F/G \to 0$.

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2. The Proof of Theorem 1.1

Let $x \ge 9$ be any real number. Let $\mathcal{S} = \{1, 2, \ldots, \lfloor\sqrt{x}\rfloor\}$. Let $\mathcal{T}$ be the set of triples (a, b, c) with $a > b > c$ all in $\mathcal{S}$. If (a, b, c) is such a triple, then

$$\max\{ab, ac, bc\} = ab < x.$$

Since the interval $[1, x]$ contains a Fibonacci number, it follows that if we write

$$ab + u = F_n, \quad ac + v = F_m, \quad bc + w = F_\ell$$

for positive integers (ℓ, m, n) such that $|u|$, $|v|$ and $|w|$ are minimal, then $\max\{|u|, |v|, |w|\} \le x$. In particular, triples (a, b, c) in $\mathcal{T}$ are counted by $f(x)$. It follows that

$$f(x) \ge \binom{\#\mathcal{T}}{3} \gg x^{3/2},$$

which takes care of the lower bound.

For the upper bound, assume that $x \ge 2$ and that (a, b, c) is a triple of integers $a > b > c \ge 1$ such that

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \le x.$$

Using (1.2), we get that

$$\exp(0.034\sqrt{\log a}) < x \qquad \text{therefore,} \qquad \log a < 900(\log x)^2.$$

It thus follows that if we write $ab + u = F_n$, where $|u| = \|ab\|_F$, then

$$F_n < a^2 + x < \exp(1800(\log x)^2) + x < \exp(2000(\log x)^2). \tag{2.1}$$

We now use the Binet formula

$$F_s = \frac{\alpha^s - \beta^s}{\alpha - \beta} \qquad \text{valid for all integers} \qquad s \ge 0, \tag{2.2}$$

where $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$. In particular, the inequality

$$F_s \ge \alpha^{s-2} \qquad \text{holds for all} \qquad s \ge 1.$$

From inequality (2.1), we get

$$\alpha^{n-2} < \exp(2000(\log x)^2)),$$

which implies that $n < 5000(\log x)^2$. The same conclusions apply to the positive indices ℓ, m such that $ac + v = F_m$, $bc + w = F_\ell$, where $|v| = \|ac\|_F$ and $|w| = \|bc\|_F$. Thus,

$$\max\{\ell, m, n\} = O((\log x)^2). \tag{2.3}$$

Since $u, v, w \in [-x, x]$, it follows that (u, v, w) can be chosen in $O(x^3)$ ways, and by inequality (2.3), the triple (ℓ, m, n) can be chosen in $O((\log x)^6)$ ways. Hence, the sextuple (ℓ, m, n, u, v, w) can be chosen in $O(x^3(\log x)^6)$ ways and once these data are chosen then

$$ab = F_n - u, \qquad ac = F_m - v \qquad \text{and} \qquad bc = F_{\ell-w},$$

therefore a, b, and c are uniquely determined. This argument shows that $f(x) \ll x^3(\log x)^6$. We shall now improve this to $f(x) \le x^{2+o(1)}$ as $x \to \infty$.

We distinguish two cases.

Case 1. $a < x^{10}$.

In this case, we fix (u, v, n, m). This can be done in $O(x^2(\log x)^4)$ ways. Once these are fixed, then

$$ab = F_n - u, \qquad \text{and} \qquad ac = F_m - v$$

are fixed. Clearly, $ab < a^2 + x < 2x^{20}$. Thus, a is a divisor of the number $ab = F_n - u$ which is of size $O(x^{20})$, so the number of choices for a is at most $\tau(F_n - u) = x^{o(1)}$ as $x \to \infty$. Here, $\tau(m)$ is the number of divisors of the positive integer m. Once a is determined, also b and c are determined out of knowledge of ab and ac. Hence, the number of triples $a > b > c \geq 1$ in this case is at most $x^{2+o(1)}$ as $x \to \infty$, which is what we wanted.

Case 2. $a \geq x^{10}$.

Fix (u, v, ℓ, m, n). This can be done in $O(x^2(\log x)^6)$ ways. Let $D = \gcd(ab, ac)$ and let $ab = Db_0$, $ac = Dc_0$. Then $a \mid D$, so we let $D = ad$. Thus, $b = b_0 d$ $c = c_0 d$. Clearly, b_0 and c_0 are uniquely determined in terms of ab and ac, so it remains to account for the number of choices for d. Observe that

$$\frac{b_0}{c_0} = \frac{ab}{ac} = \frac{F_n - u}{F_m - v}, \qquad \text{so} \qquad b_0 F_m - c_0 F_n = b_0 v - c_0 u.$$

Writing F_m and F_n according to the Binet formula (2.2), we have

$$\alpha^m (b_0 - c_0 \alpha^{n-m}) = \sqrt{5}(b_0 v - c_0 u) + \beta^m b_0 - \beta^n c_0. \tag{2.4}$$

Observe that

$$F_m = ab - u \geq ab - x > ac + x \geq F_n,$$

where the middle inequality follows because $ab - ac = a(b - c) \geq a > x^{10} > 2x$. Thus, $m < n$. The number $b_0 - c_0\alpha^{n-m}$ is a quadratic integer in $\mathbb{Q}[\sqrt{5}]$ which is not zero because if it were, then $\alpha^{n-m} = b_0/c_0 \in \mathbb{Q}$, which is impossible for $n > m$. The conjugate of $b_0 - c_0\alpha^{n-m}$ is $b_0 - c_0\beta^{n-m}$ and so

$$|b_0 - c_0\alpha^{n-m}||b_0 - c_0\beta^{n-m}| \geq 1.$$

Inserting the above inequality into (2.4) leads to

$$\alpha^m < |\sqrt{5}(b_0 v - c_0 u) + \beta^m b_0 - \beta^n c_0||b_0 - c_0\beta^{n-m}| \ll b_0^2 x.$$

Since

$$\alpha^m > F_m = ac + v \geq ac - x \geq a/2,$$

we get that $a \ll b_0^2 x$. Thus, $x^{10} \leq a \ll b_0^2 x$, therefore $b_0 \gg x^{4.5}$. We now look at the condition

$$bc + w = F_\ell,$$

which we write under the form

$$w = F_\ell - bc = F_\ell - b_0 c_0 d^2.$$

We show that there is at most one d such that $F_\ell - b_0 c_0 d^2 = w \in [-x, x]$ for large x. Assume that there were two such d, let us call them $d_1 < d_2$. Then

$$F_\ell - b_0 c_0 d_1^2 = w_1, \qquad F_\ell - b_0 c_0 d_2^2 = w_2$$

and both $w_1, w_2 \in [-x, x]$. Taking the difference of the above relations, we get

$$b_0 c_0 (d_2 - d_1)(d_2 + d_1) = (F_\ell - b_0 c_0 d_1^2) - (F_\ell - b_0 c_0 d_2^2) = w_1 - w_2 \in [-2x, 2x],$$

which is impossible for $x > x_0$ because the integer on the left above is nonzero and divisible by $b_0 c_0 \geq b_0 \gg x^{4.5}$, while the integer on the right is of absolute value at most $4x$. This shows that for large x, the quintuple (u, v, ℓ, m, n) determines d (hence, w) uniquely (at most), so the number of possible triples $a > b > c \geq 1$ in this case is $O(x^2(\log x)^6) = O(x^{2+o(1)})$ as $x \to \infty$.

The upper bound from the theorem now follows.

3. The Proof of Theorem 1.2

Consider the function (1.3) if $x = 2$. A computer search provides the results of the theorem. To turn to the details, first let

$$ab + u = F_n, \qquad ac + v = F_m, \qquad bc + w = F_\ell. \tag{3.1}$$

The condition $a < \exp(415.62)$ comes from Theorem 1 of [1]. Consequently, $n \leq 1730$ since the inequalities $\alpha^{n-2} < F_n < a^2$ hold. Then, we apply a computer search for checking all the candidates (n, m, ℓ).

We found 222 solutions $(a, b, c, u, v, w, n, m, \ell)$ to the system (3.1) with $|u|, |v|, |w| \leq 2$ belonging to 49 triples (a, b, c). Therefore $f(2) = 49$. Among the aforementioned 222 solutions, there are 43 for which $|u|, |v|, |w| \leq 1$ (see Table 1). The rows signed by * mean two solutions since $F_1 = F_2 = 1$. Concentrating only on the triples (a, b, c) again, we get $f(1) = 16$. Finally, we note that $f(0) = 0$.

4. Comments and an Open Problem

As the referee noted, the upper bound in Theorem 1.1 on $f(x)$ remains valid if we replace the Fibonacci sequence $\{F_n\}_{n\geq 0}$ with any sequence $\mathbf{u} = \{u_n\}_{n\geq 0}$ and then define $\|x\|_{\mathbf{u}}$ and $f(x)$ in ways analogously to (1.1) and (1.2), respectively. We thank the referee for this observation. Theorem 1.1 shows that

$$\frac{3}{2} \leq \liminf_{x\to\infty} \frac{\log f(x)}{\log x} \leq \limsup_{x\to\infty} \frac{\log f(x)}{\log x} \leq 2.$$

We conjecture that $\log f(x)/\log x$ tends to $3/2$ as $x \to \infty$, and we leave this as an open question for the reader.

References

[1] F. Luca and L. Szalay, *On the Fibonacci distances of ab, ac and bc*, Annales Mathematicae et Informaticae, **41** (2013), 137–163.

	a	b	c	u	v	w	F_n	F_m	F_ℓ
1	3	2	1	-1	-1	-1	5	2	$1^\star$
2	3	2	1	-1	-1	0	5	2	2
3	3	2	1	-1	-1	1	5	2	3
4	3	2	1	-1	0	-1	5	3	$1^\star$
5	3	2	1	-1	0	0	5	3	2
6	3	2	1	-1	0	1	5	3	3
7	4	2	1	0	-1	-1	8	3	$1^\star$
8	4	2	1	0	-1	0	8	3	2
9	4	2	1	0	-1	1	8	3	3
10	4	2	1	0	1	-1	8	5	$1^\star$
11	4	2	1	0	1	0	8	5	2
12	4	2	1	0	1	1	8	5	3
13	4	3	1	1	-1	-1	13	3	2
14	4	3	1	1	-1	0	13	3	3
15	4	3	1	1	1	-1	13	5	2
16	4	3	1	1	1	0	13	5	3
17	4	3	2	1	0	-1	13	8	5
18	5	4	1	1	0	-1	21	5	3
19	5	4	1	1	0	1	21	5	5
20	6	2	1	1	-1	-1	13	5	$1^\star$
21	6	2	1	1	-1	0	13	5	2
22	6	2	1	1	-1	1	13	5	3
23	7	2	1	-1	1	-1	13	8	$1^\star$
24	7	2	1	-1	1	0	13	8	2
25	7	2	1	-1	1	1	13	8	3
26	7	3	1	0	1	-1	21	8	2
27	7	3	1	0	1	0	21	8	3
28	7	3	2	0	-1	-1	21	13	5
29	7	5	1	-1	1	0	34	8	5
30	8	7	1	-1	0	1	55	8	8
31	9	6	1	1	-1	-1	55	8	5
32	11	3	2	1	-1	-1	34	21	5
33	14	4	1	-1	-1	-1	55	13	3
34	14	4	1	-1	-1	1	55	13	5
35	22	4	1	1	-1	-1	89	21	3
36	22	4	1	1	-1	1	89	21	5
37	54	7	1	-1	1	1	377	55	8

Table 1

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