GAP BALANCING NUMBERS

G. K. PANDA AND S. S. ROUT

ABSTRACT. Gap balancing numbers are introduced and defined. It is observed that 0-gap and 1-gap balancing numbers are nothing but cobalancing and balancing numbers respectively. A detailed study of 2-gap balancing numbers is presented.

1. INTRODUCTION

In [1], Behera and Panda defined balancing numbers n and balancers r as solutions of the Diophantine Equation $1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$. Subsequently, Panda and Ray [7] introduced cobalancing numbers n and cobalancers r as solutions of the Diophantine Equation $1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$. After that, several papers came up in this area and the interested readers are advised to read [1,2] and [4–11] for a literature review. Observe that while defining balancing numbers, we delete a number (and hence maintain a gap) from the list of first m natural numbers so that, the sum of numbers to the left of it is equal to the sum to the right. In case of cobalancing numbers, there is no such gap. To define gap balancing numbers, we shall consider deleting $k < m - 2$ consecutive numbers from the list of first m natural numbers so that the sum of numbers to the left of these deleted numbers is equal to the sum to the right. In this paper, we focus our attention on 2-gap balancing numbers only.

2. k-GAP BALANCING NUMBERS

In this section, we define k-gap balancing numbers and provide some examples.

Definition 2.1. *Let k be an odd natural number. We call a natural number n a k-gap balancing number (or g_k-balancing number) if*

$$1 + 2 + \cdots + \left(n - \frac{k+1}{2}\right) = \left(n + \frac{k+1}{2}\right) + \left(n + \frac{k+3}{2}\right) + \cdots + (n + r)$$

for some natural number r, which we call a k-gap balancer (or a g_k-balancer) corresponding to n.

Definition 2.2. *Let k be even. If*

$$1 + 2 + \cdots + \left(n - \frac{k}{2}\right) = \left(n + \frac{k}{2} + 1\right) + \left(n + \frac{k}{2} + 2\right) + \cdots + (n + r)$$

for some natural numbers n and r then we call $2n+1$ a k-gap balancing number (or g_k-balancing number) and r a k-gap balancer (or a g_k-balancer) corresponding to this k-gap balancing number.

Since our focus in this paper is on 2-gap balancing numbers (henceforth we will call g_2-balancing numbers), we prefer to provide a formal definition of g_2-balancing numbers separately.

Definition 2.3. *We call* $2n+1$ *a* g_2*-balancing number if*

$$1+2+\cdots+(n-1)=(n+2)+(n+3)+\cdots+(n+r)$$

for some natural number r*. We call* r *the* g_2*-balancer corresponding to the* g_2*-balancing number* $2n+1$*.*

Example 2.4. *Since* $1+2+3=6,\ 9$ *is a* g_2*-balancing number with* g_2*-balancer* 2*. Similarly, since* $1+2+\cdots+8=11+12+13,\ 19$ *is a* g_2*-balancing number with* g_2*-balancer* 4*.*

Remark 2.5. *The defining equation for* g_2*-balancing numbers suggests that if* $x=2n+1$ *is a* g_2*-balancing number then*

$$r=\frac{-x+\sqrt{2x^2+7}}{2}.$$

Thus, if x *is a* g_2*-balancing number then* $2x^2+7$ *is a perfect square. It is easy to see that* 9 *is the first* g_2*-balancing number. Since* $2\cdot 1^2+7=9=3^2$ *and* $2\cdot 3^2+7=25=5^2$*, we accept 1 and 3 as* g_2*-balancing numbers (though these numbers do not satisfy the definition of* g_2*-balancing numbers), just like Behera and Panda [1] accepted* 1 *as the first balancing number and Panda and Ray [7] accepted* 0 *as the first cobalancing number. After adding* 1 *and* 3 *to* g_2*-balancing numbers' list, we can claim that a natural number* x *is a* g_2*-balancing number if and only if* $2x^2+7$ *is a perfect square.*

3. Functions Generating g_2-Balancing Numbers

In this section, we present some functions that generate g_2-balancing numbers. The following theorems contain these functions.

Theorem 3.1. *If* x *is a* g_2*-balancing number then* $f(x)=3x+2\sqrt{2x^2+7}$ *is also a* g_2*-balancing number. Furthermore,* $f(x)\equiv 1$ *or* $-1 (mod\ 4)$ *according to the* g_2*-balancing number* $x\equiv 1$ *or* $-1(mod\ 4)$*.*

Proof. The identity

$$2f^2(x)+7=\left(4x+3\sqrt{2x^2+7}\right)^2$$

together with Remark 2.5 proves that $f(x)$ is a g_2-balancing number. We observe that $2x^2+7\equiv 1 \pmod 4$ if $x\equiv \pm 1 \pmod 4$ and hence, $\sqrt{2x^2+7}\equiv \pm 1 \pmod 4$. If $x\equiv 1 \pmod 4$, then

$$3x+2\sqrt{2x^2+7}\equiv 3\cdot 1\pm 2 \pmod 4 \equiv 1 \pmod 4$$

and if $x\equiv -1 \pmod 4$, then

$$3x+2\sqrt{2x^2+7}\equiv 3\cdot(-1)\pm 2 \pmod 4 \equiv -1 \pmod 4.$$

□

Theorem 3.2. *If* x *is a* g_2*-balancing number and* $x\equiv -1 \pmod 4$*, then* $g(x)=\frac{11x+6\sqrt{2x^2+7}}{7}$ *is also a* g_2*-balancing number and* $g(x)\equiv 1 \pmod 4$*.*

Proof. We first show that if x is a g_2-balancing number and $x\equiv -1 \pmod 4$, then $g(x)$ is a natural number, that is,

$$11x+6\sqrt{2x^2+7}\equiv 0 \pmod 7. \tag{3.1}$$

Since $2x^2+7\equiv 9x^2 \pmod 7$, it follows that $\sqrt{2x^2+7}\equiv \pm 3x \pmod 7$. This gives

$$11x+6\sqrt{2x^2+7}\equiv 11x\pm 18x \pmod 7,$$

implying that

$$11x + 6\sqrt{2x^2+7} \equiv -7x \pmod 7 \quad \text{or} \quad 11x + 6\sqrt{2x^2+7} \equiv 29x \pmod 7.$$

Thus,

$$11x + 6\sqrt{2x^2+7} \equiv 0 \pmod 7$$

or

$$10x + 6\sqrt{2x^2+7} \equiv 0 \pmod 7. \tag{3.2}$$

Observe that the substitution $x = 3$ (and hence, $x \equiv -1 \pmod 4$) in (3.2) yields $4 \equiv 0 \pmod 7$ which is false. Thus, the only option left is $11x + 6\sqrt{2x^2+7} \equiv 0 \pmod 7$, proving that $g(x)$ is a natural number if x is a g_2-balancing number and $x \equiv -1 \pmod 4$. Now, by virtue of Remark 2.5 and the identity

$$2g^2(x) + 7 = \left(\frac{12x + 11\sqrt{2x^2+7}}{7}\right)^2,$$

$g(x)$ is a g_2-balancing number. Finally, we have to show that $g(x) \equiv 1 \pmod 4$. We observe that

$$g(x) \equiv (-1)\cdot(11x + 6\sqrt{2x^2+7}) \pmod 4$$

since $7^{-1} \equiv -1 \pmod 4$. Thus, if $x \equiv -1 \pmod 4$ then $g(x) \equiv -1 \pm 6 \equiv 1 \pmod 4$. This ends the proof. ☐

Theorem 3.3. *If x is a g_2-balancing number and $x \equiv 1 \pmod 4$, then $h(x) = \frac{9x+4\sqrt{2x^2+7}}{7}$ is also a g_2-balancing number and $h(x) \equiv -1 \pmod 4$.*

Proof. First of all we claim that $h(x)$ is a natural number. For this, we have to show that if x is a g_2-balancing number and $x \equiv 1 \pmod 4$, then

$$9x + 4\sqrt{2x^2+7} \equiv 0 \pmod 7. \tag{3.3}$$

Since $2x^2 + 7 \equiv 9x^2 \pmod 7$, it follows that $\sqrt{2x^2+7} \equiv \pm 3x \pmod 7$. Hence,

$$9x + 4\sqrt{2x^2+7} \equiv 9x \pm 12x \pmod 7,$$

which gives

$$9x + 4\sqrt{2x^2+7} \equiv 21x \pmod 7 \quad \text{or} \quad 9x + 4\sqrt{2x^2+7} \equiv -3x \pmod 7.$$

Thus either

$$9x + 4\sqrt{2x^2+7} \equiv 0 \pmod 7$$

or

$$12x + 4\sqrt{2x^2+7} \equiv 0 \pmod 7. \tag{3.4}$$

But the substitution $x = 1$ (and hence, $x \equiv 1 \pmod 4$) in (3.4) gives $3 \equiv 0 \pmod 7$ which is not true. Thus the only option left is $9x + 4\sqrt{2x^2+7} \equiv 0 \pmod 7$, proving that $h(x)$ is a natural number if x is a g_2-balancing number and $x \equiv 1 \pmod 4$. Our next claim is that $h(x)$ is a g_2-balancing number. This easily follows from the identity

$$2h^2(x) + 7 = \left(\frac{8x + 9\sqrt{2x^2+7}}{7}\right)^2$$

and Remark 2.5. Lastly, it remains to show that $h(x) \equiv -1 \pmod 4$. Since

$$h(x) \equiv (-1)\cdot(9x + 4\sqrt{2x^2+7}) \pmod 4$$

and $x \equiv 1 \pmod 4$ it follows that $h(x) \equiv -1 \pmod 4$. This ends the proof. $\square$

4. Listing All g_2-Balancing Numbers

In the last section, we presented some functions that generate g_2-balancing numbers from the given ones. Indeed we have seen in Remark 2.5 that x is a g_2-balancing number if and only if $2x^2+7$ is a perfect square. In this section, we solve the Diophantine Equation $2x^2+7=y^2$ and provide the list of all g_2-balancing numbers. Of course, the method of solving $2x^2+7=y^2$ is not direct, rather we convert $2x^2+7=y^2$ to a Pells' equation of the form $8z^2+1=w^2$ and apply certain balancing numbers' treatment (see [1, p. 98]).

Let x be any g_2-balancing number so that $2x^2+7$ is a perfect square. Now the congruence $9x^2 \equiv 2x^2+7 \pmod 7$ gives $3x \equiv \pm\sqrt{2x^2+7} \pmod 7$. Since both x and $2x^2+7$ are odd, we also have

$$3x \equiv \pm\sqrt{2x^2+7} \pmod 2.$$

Thus $3x \pm \sqrt{2x^2+7}$ is congruent to 0 modulo 2 and modulo 7. As 2 and 7 are coprimes,

$$3x \pm \sqrt{2x^2+7} \equiv 0 \pmod{14},$$

yielding that either $\frac{3x+\sqrt{2x^2+7}}{14}$ or $\frac{3x-\sqrt{2x^2+7}}{14}$ is a natural number. Since

$$8 \cdot \left[\frac{3x \pm \sqrt{2x^2+7}}{14}\right]^2 + 1 = \left[\frac{3\sqrt{2x^2+7} \pm 2x}{7}\right]^2,$$

by virtue of [1, p. 98], it follows that either $\frac{3x+\sqrt{2x^2+7}}{14}$ or $\frac{3x-\sqrt{2x^2+7}}{14}$ is a balancing number. Letting

$$B = \frac{3x+\sqrt{2x^2+7}}{14} \quad \text{or} \quad B = \frac{3x-\sqrt{2x^2+7}}{14}$$

we obtain

$$(14B-3x)^2 = 2x^2+7.$$

This leads to the quadratic equation

$$x^2 - 12Bx + 28B^2 - 1 = 0.$$

The solutions of this equation are $x = 6B \pm \sqrt{8B^2+1} = 6B \pm C$ where C is the Lucas-balancing number associated with B [8]. We further observe that

$$2 \cdot (6B \pm C)^2 + 7 = (3C \pm 4B)^2.$$

Thus all the g_2-balancing numbers are given by $6B \pm C$. As usual, for $n = 0, 1, \dots$ we denote the nth balancing number by B_n and nth Lucas-balancing number by C_n [8]. Hence, $\{6B_n - C_n,\ 6B_n + C_n : n = 1, 2, \dots\}$ is the exhaustive list of all g_2-balancing numbers. We next show that for each natural number n,

$$6B_n - C_n < 6B_n + C_n < 6B_{n+1} - C_{n+1}.$$

The first part of this inequality is obvious. To prove the second part, we observe that in view of $B_{n-1} = 3B_n - C_n,\ B_{n+1} = 3B_n + C_n$ (see [8, p. 186]) and $B_n > 0$ if $n \geq 1$, it follows that

$C_n < 3B_n$ for $n > 1$. Also, we know that for each natural number $n, B_{n-1} < B_n$. Hence,

$$\begin{aligned}6B_n + C_n &= 3B_n + 3B_n + C_n = 3B_n + B_{n+1}\\ &< 11B_n + B_{n+1} + 2(B_n - B_{n-1})\\ &= 2(6B_n - B_{n-1}) + B_{n+1} + B_n\\ &= 2B_{n+1} + B_{n+1} + 3B_{n+1} - C_{n+1} = 6B_{n+1} - C_{n+1}.\end{aligned}$$

We shall denote the nth g_2-balancing number by x_n. Thus, the first g_2-balancing number is $x_1 = 6B_1 - C_1 = 6 \cdot 1 - 3 = 3$, the second one is $x_2 = 6B_1 + C_1 = 9$, the third one $x_3 = 6B_2 - C_2 = 6 \cdot 6 - 17 = 19$ and the fourth one is $x_4 = 6B_2 + C_2 = 53$ and so on. In general $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n, n = 1, 2, \ldots$. Further, we may write $x_0 = 6B_0 + C_0 = 6B_0 + \sqrt{8B_0^2 + 1} = 6 \cdot 0 + \sqrt{8 \cdot 0^2 + 1} = 1$.

The above discussion proves the following theorem.

Theorem 4.1. *If x is a g_2-balancing number then $x = 6B_n - C_n$ or $x = 6B_n + C_n$ for some natural number n. In particular, if we denote the nth g_2-balancing number by x_n, then $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n, n = 1, 2, \ldots$.*

The next theorem classifies g_2-balancing numbers congruent to 1 and -1 modulo 4.

Theorem 4.2. *For $n = 1, 2, \ldots, x_{2n-1} \equiv -1 \pmod 4$ and $x_{2n} \equiv 1 \pmod 4$.*

To prove this theorem, we need the following lemma.

Lemma 4.3. *If n is even, then $6B_n \equiv 0 \pmod 4$ and $C_n \equiv 1 \pmod 4$; if n is odd, then $6B_n \equiv 2 \pmod 4$ and $C_n \equiv -1 \pmod 4$.*

Proof. We know that B_n is even or odd when n is even or odd, respectively. Therefore, if n is even then $6B_n \equiv 0 \pmod 4$. Further, if n is odd, then B_n is odd and $B_n \equiv \pm 1 \pmod 4$ implies $6B_n \equiv \pm 6 \equiv 2 \pmod 4$. Further $C_1 = 3 \equiv -1 \pmod 4$ and $C_2 = 17 \equiv 1 \pmod 4$. Assume that $C_{2n-1} \equiv -1 \pmod 4$ and $C_{2n} \equiv 1 \pmod 4$ for $n = 1, 2, \ldots, k$. Then

$$C_{2k+1} = 6C_{2k} - C_{2k-1} \equiv 6 \cdot 1 - (-1) \equiv -1 \pmod 4$$

and

$$C_{2k+2} = 6C_{2k+1} - C_{2k} \equiv 6 \cdot (-1) - 1 \equiv 1 \pmod 4.$$

□

Proof of Theorem 4.2. We infer from Lemma 4.3 that if n is even then $6B_n \equiv 0 \pmod 4$ and $C_n \equiv 1 \pmod 4$. Hence, $6B_n + C_n \equiv 0 + 1 \equiv 1 \pmod 4$ and $6B_n - C_n \equiv 0 - 1 \equiv -1 \pmod 4$. Similarly, if n is odd $6B_n + C_n \equiv 2 + (-1) \equiv 1 \pmod 4$ and $6B_n - C_n \equiv 2 + 1 \equiv -1 \pmod 4$. Thus, $x_{2n-1} \equiv -1 \pmod 4$ and $x_{2n} \equiv 1 \pmod 4$, $n = 1, 2, \ldots$. □

5. Recurrence Relations for g_2-Balancing Numbers

In the previous section we have seen that, g_2-balancing numbers are given by $x_{2n-1} = 6B_n - Cn$ and $x_{2n} = 6B_n + C_n$, $n = 1, 2, \ldots$. Since both balancing as well as Lucas-balancing numbers satisfy the recurrence relation $y_{n+1} = 6y_n - y_{n-1}$ ([1, p. 100] and [9, p. 44]), it follows that the g_2-balancing numbers satisfy the recurrence relation $x_{n+2} = 6x_n - x_{n-2}$; $n = 3, 4, \ldots$.

In Section 3, we developed some non-linear functions for finding a specified type of g_2-balancing numbers from the given ones. Here we shall prove that two of these functions are nothing but shift functions to the next g_2-balancing numbers. In this context we have the following theorems.

Theorem 5.1. *Let* $g(x) = \frac{11x+6\sqrt{2x^2+7}}{7}$ *and* $h(x) = \frac{9x+4\sqrt{2x^2+7}}{7}$ *be two arithmetic functions. Then* $g(x_{2n-1}) = x_{2n}$ *and* $h(x_{2n}) = x_{2n+1}$.

Proof. If $x = 6B_n \pm C_n$, then $2x^2 + 7 = (3C_n \pm 4B_n)^2$. Thus if $x = x_{2n-1}$, then

$$\begin{aligned}
g(x_{2n-1}) &= \frac{11(6B_n - C_n) + 6\sqrt{2(6B_n - C_n)^2 + 7}}{7} \\
&= \frac{11(6B_n - C_n) + 6(3C_n - 4B_n)}{7} \\
&= 6B_n + C_n = x_{2n},
\end{aligned}$$

and for $x = x_{2n}$

$$\begin{aligned}
h(x_{2n}) &= \frac{9(6B_n + C_n) + 4\sqrt{2(6B_n + C_n)^2 + 7}}{7} \\
&= \frac{9(6B_n + C_n) + 4(3C_n + 4B_n)}{7} \\
&= 10B_n + 3C_n = 3(3B_n + C_n) + B_n \\
&= 3B_{n+1} + (3B_{n+1} - C_{n+1}) \\
&= 6B_{n+1} - C_{n+1} = x_{2n+1}.
\end{aligned}$$

□

It is important to observe that $g(x) = \frac{11x+6\sqrt{2x^2+7}}{7}$ and $h(x) = \frac{9x+4\sqrt{2x^2+7}}{7}$ are strictly increasing functions for $x > 0$. So the functions are invertible. It is easy to see that $g^{-1}(y) = \frac{11y-6\sqrt{2y^2+7}}{7}$ and $h^{-1}(y) = \frac{9y-4\sqrt{2y^2+7}}{7}$. Thus, we can definitely expect $g^{-1}(x_{2n}) = x_{2n-1}$ and $h^{-1}(x_{2n+1}) = x_{2n}$. The following corollary demonstrates this result.

Corollary 5.2. *Let* $\tilde{g}(x) = \frac{11x-6\sqrt{2x^2+7}}{7}$ *and* $\tilde{h}(x) = \frac{9x-4\sqrt{2x^2+7}}{7}$ *be two arithmetic functions. Then* $\tilde{g}(x_{2n}) = x_{2n-1}$ *and* $\tilde{h}(x_{2n+1}) = x_{2n}$.

Proof. It is known that if $x = 6B_n \pm C_n$, then $2x^2 + 7 = (3C_n \pm 4B_n)^2$. Thus if $x = x_{2n}$, then

$$\begin{aligned}
\tilde{g}(x_{2n}) &= \frac{11(6B_n + C_n) - 6\sqrt{2(6B_n + C_n)^2 + 7}}{7} \\
&= \frac{11(6B_n + C_n) - 6(3C_n + 4B_n)}{7} \\
&= 6B_n - C_n = x_{2n-1},
\end{aligned}$$

and for $x = x_{2n+1}$

$$\begin{aligned}
\tilde{h}(x_{2n+1}) &= \frac{9(6B_{n+1} - C_{n+1}) - 4\sqrt{2(6B_{n+1} - C_{n+1})^2 + 7}}{7} \\
&= \frac{9(6B_{n+1} - C_{n+1}) - 4(3C_{n+1} - 4B_{n+1})}{7} \\
&= 10B_{n+1} - 3C_{n+1} = 3(3B_{n+1} - C_{n+1}) + B_{n+1} \\
&= 3B_n + (3B_n + C_n) \\
&= 6B_n + C_n = x_{2n}.
\end{aligned}$$

□

Theorem 5.1 gives functions shifting g_2-balancing numbers to the next ones, while Corollary 5.2 provide functions taking g_2-balancing numbers to previous ones. In the following theorem, we introduce a function which transforms an odd order g_2-balancing number to the next odd order g_2-balancing number and also an even order g_2-balancing number to the next even order g_2-balancing number.

Theorem 5.3. *Let $f(x) = 3x + 2\sqrt{2x^2+7}$ be an arithmetic function. Then $f(x_n) = x_{n+2}$.*

Proof. Here also we shall use the fact that if $x = 6B_n \pm C_n$, then $2x^2 + 7 = (3C_n \pm 4B_n)^2$. Now,

$$\begin{aligned} f(x_{2n-1}) &= 3x_{2n-1} + 2\sqrt{2x_{2n-1}^2 + 7} \\ &= 3(6B_n - C_n) + 2(3C_n - 4B_n) \\ &= 10B_n + 3C_n. \end{aligned}$$

In the proof of Theorem 5.1, it has been shown that $10B_n + 3C_n = x_{2n+1}$. Further,

$$\begin{aligned} f(x_{2n}) &= 3x_{2n} + 2\sqrt{2x_{2n}^2 + 7} \\ &= 3(6B_n + C_n) + 2(3C_n + 4B_n) \\ &= 26B_n + 9C_n = 9(3B_n + C_n) - B_n \\ &= 9B_{n+1} - B_n = 6B_{n+1} + 3B_{n+1} - B_n \\ &= 6B_{n+1} + C_{n+1} = x_{2n+2}. \end{aligned}$$

□

It is important to note that $f(x) = 3x + 2\sqrt{2x^2+7}$ is strictly increasing for $x > 0$. So the inverse exists and it is easy to see that $f^{-1}(y) = 3y - 2\sqrt{2y^2+7}$. Thus, we can definitely expect $f^{-1}(x_{2n}) = x_{2n-2}$ and $f^{-1}(x_{2n+1}) = x_{2n-1}$. The following corollary ascertains these claims.

Corollary 5.4. *Let $\tilde{f}(x) = 3x - 2\sqrt{2x^2+7}$ be an arithmetic function. Then $\tilde{f}(x_n) = x_{n-2}$.*

The proof of this corollary is similar to that of Theorem 5.3 and hence it is omitted.

6. Binet Form for g_2-Balancing Numbers

In Section 5, we obtained the recurrence relation $x_{n+2} = 6x_n - x_{n-2}$ for g_2-balancing numbers, which is linear, homogeneous and is of fourth order. Using this recurrence relation, we can find the Binet form (also popularly known as closed form) for g_2-balancing numbers.

Putting $x_n = \alpha^n$ as a trial solution in $x_{n+2} = 6x_n - x_{n-2}$ we get the auxiliary equation $\alpha^4 - 6\alpha^2 + 1 = 0$. The solutions of this biquadratic equation are

$$\alpha_1 = 1 + \sqrt{2}, \quad \alpha_2 = 1 - \sqrt{2}, \quad \alpha_3 = -(1 + \sqrt{2}), \quad \alpha_4 = -(1 - \sqrt{2}).$$

Hence, the general solution of $x_{n+2} = 6x_n - x_{n-2}$ is given by

$$x_n = A\alpha_1^n + B\alpha_2^n + C\alpha_3^n + D\alpha_4^n$$

and the initial conditions are $x_0 = 1,\ x_1 = 3,\ x_2 = 9$, and $x_3 = 19$. Since $\alpha_3 = -\alpha_1$ and $\alpha_4 = -\alpha_2$, it follows that

$$x_n = (A + (-1)^n C)\alpha_1^n + (B + (-1)^n D)\alpha_2^n.$$

Substitution of initial conditions yields

$$x_n = \begin{cases} \frac{\alpha_1^{n+2}-\alpha_2^{n+2}}{2\sqrt{2}} - \frac{\alpha_1^n+\alpha_2^n}{2} & \text{if } n \text{ is even,} \\ \frac{\alpha_1^{n+2}-\alpha_2^{n+2}}{2\sqrt{2}} - \frac{\alpha_1^n-\alpha_2^n}{\sqrt{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Using this result and the Binet form of balancing and Lucas-balancing numbers (see [1], [8]) we get,

$$\begin{aligned} x_{2k} &= \frac{\alpha_1^{2k+2}-\alpha_2^{2k+2}}{2\sqrt{2}} - \frac{\alpha_1^{2k}+\alpha_2^{2k}}{2} \\ &= 2\cdot\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{4\sqrt{2}} - \frac{\lambda_1^k+\lambda_2^k}{2} \\ &= 2B_{k+1} - C_k = 2(3B_k + C_k) - C_k \\ &= 6B_k + C_k, \end{aligned}$$

and

$$\begin{aligned} x_{2k-1} &= \frac{\alpha_1^{2k+1}-\alpha_2^{2k+1}}{2\sqrt{2}} - \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{\sqrt{2}} \\ &= \frac{(1+\sqrt{2})\alpha_1^{2k}-(1-\sqrt{2})\alpha_2^{2k}}{2\sqrt{2}} + \frac{(1-\sqrt{2})\alpha_1^{2k}-(1+\sqrt{2})\alpha_2^{2k}}{\sqrt{2}} \\ &= 2B_k + C_k + 4B_k - 2C_k \\ &= 6B_k - C_k \end{aligned}$$

which are already obtained in Section 4.

7. Functions Transforming g_2-Balancing Numbers to Balancing and Related Numbers

Here we present some functions of g_2-balancing numbers that generate balancing numbers. The following theorems are important in this regard.

Theorem 7.1. *If x is an odd ordered g_2-balancing number then, $F(x) = \frac{3x+\sqrt{2x^2+7}}{14}$ is a balancing number. Further, if x is an even ordered g_2-balancing number then, $\tilde{F}(x) = \frac{3x-\sqrt{2x^2+7}}{14}$ is a balancing number. In particular, $F(x_{2n-1}) = \tilde{F}(x_{2n}) = B_n$.*

Proof. Since $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n$, we have

$$F(x_{2n-1}) = \frac{3(6B_n - C_n) + 3C_n - 4B_n}{14} = B_n$$

and

$$\tilde{F}(x_{2n}) = \frac{3(6B_n + C_n) - (3C_n + 4B_n)}{14} = B_n.$$

□

The next theorem relates functions of g_2-balancing numbers to Lucas-balancing numbers.

Theorem 7.2. *If x is an odd ordered g_2-balancing number then $G(x) = \frac{2x+3\sqrt{2x^2+7}}{7}$ is a Lucas-balancing number. Further, if x is an even ordered g_2-balancing number then $\tilde{G}(x) = \frac{-2x+3\sqrt{2x^2+7}}{7}$ is a Lucas-balancing number. In particular, $G(x_{2n-1}) = \tilde{G}(x_{2n}) = C_n$.*

Proof. Since $x_{2n-1} = 6B_n - C_n$ and $x_{2n} = 6B_n + C_n$, we have

$$G(x_{2n-1}) = \frac{2(6B_n - C_n) + 3(3C_n - 4B_n)}{7} = C_n,$$

and

$$\tilde{G}(x_{2n}) = \frac{-2(6B_n + C_n) + 3(3C_n + 4B_n)}{7} = C_n.$$

□

8. An Application of g_2-Balancing Numbers to an Almost Pythagorean Equation

The association of balancing and cobalancing numbers with the solutions of Pythagorean and Pythagorean-like equations is well-known (see [1, p. 104], [7, p. 1199] and [9, p. 69]). In [3], Haggard developed certain links of solutions of the Pythagorean equation $x^2 + y^2 = z^2$ with the solutions of the almost Pythagorean equation $x^2 + y^2 = z^2 + 1$. In this section, we completely solve the Diophantine equation $x^2 + (x + 4)^2 = y^2 + 1$.

We observe that if $x^2 + (x + 4)^2 = y^2 + 1$, then x must be odd. Setting $z = x + 2$, we convert this equation to $(z-2)^2 + (z+2)^2 = y^2 + 1$, which on simplification gives $2z^2 + 7 = y^2$, suggesting that z is a g_2-balancing number, so that $z = x_n$ for some n. Now we can list the solutions of the equation $x^2 + (x + 4)^2 = y^2 + 1$ as $x = x_n - 2,\ y = \sqrt{2x_n^2 + 7}, n = 1, 2, \ldots$.

Observe that almost Pythagorean equations corresponding to the g_2-balancing number 9 and 13 are respectively, $7^2 + 11^2 = 13^2 + 1$ and $17^2 + 21^2 = 27^2 + 1$.

9. Acknowledgement

It is a pleasure to thank the anonymous referee for his valuable suggestions and comments which resulted in an improved presentation of this paper.

References

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers*, The Fibonacci Quarterly, **37.2** (1999), 98–105.
[2] A. Bérczes, K. Liptai, and I. Pink, *On generalized balancing sequences*, The Fibonacci Quarterly, **48.2** (2010), 121–128.
[3] P. W. Haggard, *Pythagorean triples and sums of triangular numbers*, Int. J. Math. Educ. Sci. Technol., **28.1** (1997), 109–116.
[4] R. Keskin and O. Karaath, *Some new properties of balancing numbers and square triangular numbers*, Journal of Integer Sequences, **15** (2012), Article 12.1.4.
[5] K. Liptai, *Fibonacci balancing numbers*, The Fibonacci Quarterly, **42.4** (2004), 330–340.
[6] K. Liptai, F. Luca, Á. Pintér, and L. Szalay, *Generalized balancing numbers*, Indag. Mathem. (N.S.), **20.1** (2009), 87–100.
[7] G. K. Panda and P. K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sci., **8** (2005), 1189–1200.
[8] G. K. Panda, *Some fascinating properties of balancing numbers*, Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications, Cong Numerantium, **194** (2009), 185–189.
[9] G. K. Panda and P. K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math. Acad. Sinica(N.S.), **6.1** (2011), 41–72.
[10] G. K. Panda and S. S. Rout, *A class of recurrent sequences exhibiting some exciting properties of balancing numbers*, Int. J. Math. Comp. Sci., **6** (2012), 4–6.
[11] T. Szakács, *Multiplying balancing numbers*, Acta Univ Sapientiae Matem, **3.1** (2011), 90–96.

MSC2010: 11B39, 11B83

Department of Mathematics, National Institute of Technology Rourkela, Odisha, 769 008, India
E-mail address: gkpanda_nit@rediffmail.com

Department of Mathematics, National Institute of Technology Rourkela, Odisha, 769 008, India
E-mail address: sudhansumath@yahoo.com