DERIVATIONS AND IDENTITIES FOR FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. We introduce the notion of Fibonacci and Lucas derivations in the algebra of polynomials. We prove that any element of the kernel of the derivations gives a polynomial identity satisfied by the Fibonacci and Lucas polynomials. Also, we prove that any polynomial identity satisfied by the Appel polynomials yields a polynomial identity satisfied by the Fibonacci and Lucas polynomials. We describe the corresponding intertwining maps.

1. Introduction

The Fibonacci $F_n(x)$ and Lucas $L_n(x)$ polynomials are defined by the following ordinary generating functions

$$\mathcal{G}(F_n(x), t) = \frac{t}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_n(x) t^n,$$

$$\mathcal{G}(L_n(x), t) = \frac{1 + t^2}{1 - xt - t^2} = \sum_{n=0}^{\infty} L_n(x) t^n.$$

The derivatives of the polynomials can be expressed in terms of the polynomials as follows

$$\frac{d}{dx} F_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) F_{n-1-2k}(x), \tag{1.1}$$

$$\frac{d}{dx} L_n(x) = n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k L_{n-1-2k}(x), \tag{1.2}$$

see [2, 3].

We are interested in finding polynomial identities satisfied by the polynomials, i.e., identities of the form

$$P(F_0(x), F_1(x), \ldots, F_n(x)) = \text{const.}, \quad \text{and } P(L_0(x), L_1(x), \ldots, L_n(x)) = \text{const.},$$

where $P(x_0, x_1, \ldots, x_n)$ is a polynomial of $n + 1$ variables.

We provide a method for finding such identities that is based on the simple observation: if

$$\frac{d}{dx} P(F_0(x), F_1(x), \ldots, F_n(x)) = 0, \tag{1.3}$$

then $P(F_0(x), F_1(x), \ldots, F_n(x))$ is constant. In other words, $P(F_0(x), F_1(x), \ldots, F_n(x))$ gives an identity of the Fibonacci polynomials.

To establish (1.3), we rewrite the derivative as:

$$\frac{d}{dx}P(F_0(x),F_1(x),\dots,F_n(x))$$
$$=\frac{\partial}{\partial x_0}P(x_0,x_1,\dots,x_n)\Big|_{\{x_i=F_i(x)\}}\frac{d}{dx}F_0(x)+\cdots+\frac{\partial}{\partial x_n}P(x_0,x_1,\dots,x_n)\Big|_{\{x_i=F_i(x)\}}\frac{d}{dx}F_n(x).$$

We have

$$\frac{d}{dx}P(F_0(x),F_1(x),\dots,F_n(x))$$
$$=\left(\frac{\partial}{\partial x_0}P(x_0,x_1,\dots,x_n)\mathcal{D}_{\mathcal{F}}(x_0)+\cdots+\frac{\partial}{\partial x_n}P(x_0,x_1,\dots,x_n)\mathcal{D}_{\mathcal{F}}(x_n)\right)\Big|_{\{x_i=F_i(x)\}}$$
$$=\mathcal{D}_{\mathcal{F}}(P(x_0,x_1,\dots,x_n))\big|_{\{x_i=F_i(x)\}},$$

where the differential operator $\mathcal{D}_{\mathcal{F}}$ is defined by

$$\mathcal{D}_{\mathcal{F}}:=x_1\frac{\partial}{\partial x_2}+2x_2\frac{\partial}{\partial x_3}+(3\,x_3-x_1)\frac{\partial}{\partial x_4}+\cdots+\left(\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(n-1-2k)x_{n-1-2k}\right)\frac{\partial}{\partial x_n}.$$

It is clear that if $\mathcal{D}_{\mathcal{F}}(P(x_0,x_1,\dots,x_n))=0$ then $\frac{d}{dx}P(F_0(x),F_1(x),\dots,F_n(x))=0$. Thus, any non-trivial polynomial $P(x_0,x_1,\dots,x_n)$, which belongs to the kernel of $\mathcal{D}_{\mathcal{F}}$ defines a polynomial identity related to the Fibonacci polynomials.

We have a similar construction for the Lucas polynomials. We introduce the differential operator

$$\mathcal{D}_{\mathcal{L}}(x_n)=n\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k x_{n-1-2k},$$

and show that the condition

$$\mathcal{D}_{\mathcal{L}}(P(x_0,x_1,\dots,x_n))=0,$$

defines a polynomial identity related to the Lucas polynomials.

We have thus proved the following theorem.

Theorem 1.1. *Let $P(x_0,x_1,\dots,x_n)$ be a polynomial.*

(i) *If $\mathcal{D}_{\mathcal{F}}(P(x_0,x_1,\dots,x_n))=0$ then $P(F_0(x),F_1(x),\dots,F_n(x))=\text{const.}$;*
(ii) *If $\mathcal{D}_{\mathcal{L}}(P(x_0,x_1,\dots,x_n))=0$ then $P(L_0(x),L_1(x),\dots,L_n(x))=\text{const.}$*

As an example, consider the polynomial $x_1x_3-x_2^2$. It is easily verified that $\mathcal{D}_{\mathcal{F}}(x_1x_3-x_2^2)=0$. Thus, $F_1(x)F_3(x)-F_2(x)^2$ is a constant. In fact,

$$F_1(x)F_3(x)-F_2(x)^2=1.$$

A similar problem was solved by the present author for the Appel polynomials [1]. There the differential operator

$$\mathcal{D}_{\mathcal{A}}=x_0\frac{\partial}{\partial x_1}+2x_1\frac{\partial}{\partial x_2}+\cdots+nx_{n-1}\frac{\partial}{\partial x_n},$$

was defined and it was shown that every element of the kernel of $\mathcal{D}_{\mathcal{A}}$ defines some polynomial identity for the Appel polynomials. Recall that a collection of polynomials $\{A_n(x)\}$, $\deg(A_n(x))=n$ are called Appel polynomials if

$$A_n'(x)=nA_{n-1}(x),n=0,1,2,\dots. \tag{1.4}$$

In this paper, we show how polynomial identities for Appel polynomials can be used to find polynomial identities for the Fibonacci and Lucas polynomials. The method uses intertwining maps. A linear multiplicative map ψ_{AF} is called a $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{F}})$-intertwining map if the following condition holds: $\psi_{AF}\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{F}}\psi_{AF}$. Any such map induces an isomorphism from $\ker \mathcal{D}_{\mathcal{A}}$ to $\ker \mathcal{D}_{\mathcal{F}}$.

For instance, the discriminant of the polynomial (in the variables X, Y)

$$x_0X^3 + 3\,x_1X^2Y + 3\,x_2XY^2 + x_3Y^3,$$

equals

$$\begin{vmatrix} x_0 & 3x_1 & 3x_2 & x_3 & 0 \\ 0 & x_0 & 3x_1 & 3x_2 & x_3 \\ 3x_0 & 6x_1 & 3x_2 & 0 & 0 \\ 0 & 3x_0 & 6x_1 & 3x_2 & 0 \\ 0 & 0 & 3x_0 & 6x_1 & 3x_2 \end{vmatrix} = 27(6\,x_0x_3x_2x_1 + 3\,{x_1}^2{x_2}^2 - 4\,{x_1}^3x_3 - 4\,{x_2}^3x_0 - {x_0}^2{x_3}^2).$$

It is a well-known result of the classical invariant theory that the discriminant lies in the kernel of the operator $\mathcal{D}_{\mathcal{A}}$.

It is easily checked that the linear multiplicative map defined by

$$\begin{aligned} &\psi_{AL}(x_0) = x_0, \psi_{AL}(x_1) = x_1, \\ &\psi_{AL}(x_2) = x_2, \psi_{AL}(x_3) = x_3 + 3x_1, \\ &\psi_{AL}(f \cdot g) = \psi_{AL}(f) \cdot \psi_{AL}(g), f, g \in \mathbb{C}[x_0, x_1, \ldots, x_n], \end{aligned}$$

commutes with the operators $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{L}}$. Therefore the element

$$\begin{vmatrix} \psi_{AL}(x_0) & 3\psi_{AL}(x_1) & 3\psi_{AL}(x_2) & \psi_{AL}(x_3) & 0 \\ 0 & \psi_{AL}(x_0) & 3\psi_{AL}(x_1) & 3\psi_{AL}(x_2) & \psi_{AL}(x_3) \\ 3\psi_{AL}(x_0) & 6\psi_{AL}(x_1) & 3\psi_{AL}(x_2) & 0 & 0 \\ 0 & 3\psi_{AL}(x_0) & 6\psi_{AL}(x_1) & 3\psi_{AL}(x_2) & 0 \\ 0 & 0 & 3\psi_{AL}(x_0) & 6\psi_{AL}(x_1) & 3\psi_{AL}(x_2) \end{vmatrix}$$

$$= \begin{vmatrix} x_0 & 3x_1 & 3x_2 & x_3 + 3x_1 & 0 \\ 0 & x_0 & 3x_1 & 3x_2 & x_3 + 3x_1 \\ 3x_0 & 6x_1 & 3x_2 & 0 & 0 \\ 0 & 3x_0 & 6x_1 & 3x_2 & 0 \\ 0 & 0 & 3x_0 & 6x_1 & 3x_2 \end{vmatrix},$$

lies in the kernel of the operator $\mathcal{D}_{\mathcal{L}}$ and defines the following identity for the Lucas polynomial:

$$\begin{vmatrix} L_0(x) & 3L_1(x) & 3L_2(x) & L_3(x) + 3L_1(x) & 0 \\ 0 & L_0(x) & 3L_1(x) & 3L_2(x) & L_3(x) + 3L_1(x) \\ 3L_0(x) & 6L_1(x) & 3L_2(x) & 0 & 0 \\ 0 & 3L_0(x) & 6L_1(x) & 3L_2(x) & 0 \\ 0 & 0 & 3L_0(x) & 6L_1(x) & 3L_2(x) \end{vmatrix} = -864.$$

In the paper we present methods of the theory of locally nilpotent derivation to find polynomial identities for the Fibonacci and Lucas polynomials.

In Section 2, we give a brief introduction to the theory of locally nilpotent derivations. We also introduce the notion of the Fibonacci and Lucas derivations and find their kernels. In this way we obtain polynomial identities for the Fibonacci and Lucas polynomials.

In the final section, we find a $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{F}})$-intertwining map and a $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{L}})$-intertwining map.

2. Fibonacci and Lucas Derivations

2.1. **Derivations and their kernels.** Let $\mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ be the polynomial algebra in $n+1$ variables $x_0, x_1, x_2, \ldots, x_n$ over $\mathbb{C}$. Recall that a *derivation* of the polynomial algebra $\mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ is a linear map D satisfying the Leibniz rule:

$$D(f\,g) = D(f)g + fD(g), \text{ for all } f, g \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n].$$

By using the quotient rule of derivations we can extend a derivation to the field of fractions $\mathbb{C}(x_0, x_1, x_2, \ldots, x_n)$.

A derivation D is called *locally nilpotent* if for every $f \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ there is an $n \in \mathbb{N}$ such that $D^n(f) = 0$. Any derivation D is completely determined by the elements $D(x_i)$. A derivation D is called *linear* if $D(x_i)$ is a linear form. A linear locally nilpotent derivation is called a *Weitzenböck derivation.* A derivation D is called triangular if $D(x_i) \in \mathbb{C}[x_0, \ldots, x_{i-1}]$. Any triangular derivation is locally nilpotent.

The subalgebra

$$\ker D := \{f \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n] \mid D(f) = 0\},$$

is called the *kernel* of derivation D.

For an arbitrary locally nilpotent derivation D the following theorem holds.

Theorem 2.1 ([5, 6])**.** *Suppose that there exists a polynomials h such that $D(h) \neq 0$ but $D^2(h) = 0$. Then*

$$\ker D = \mathbb{C}[\sigma(x_0), \sigma(x_1), \ldots, \sigma(x_n)][D(h)^{-1}] \cap \mathbb{C}[x_0, x_1, \ldots, x_n],$$

where σ is the Diximier map

$$\sigma(x_i) = \sum_{k=0}^{\infty} D^k(x_i)\frac{\lambda^k}{k!}, \lambda = -\frac{h}{D(h)} \in \mathbb{C}(x_0, x_1, \ldots, x_n), D(\lambda) = -1.$$

The expressions (1.1) and (1.2) motivate the following definition.

Definition 1. *Derivations of $\mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ defined by*

$$D_{\mathcal{F}}(x_0) = D_{\mathcal{F}}(x_1) = 0, D_{\mathcal{F}}(x_n) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) x_{n-1-2k},$$

$$D_{\mathcal{L}}(x_0) = 0, D_{\mathcal{L}}(x_n) = n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k x_{n-1-2k},$$

are called **the Fibonacci derivation** *and* **the Lucas derivation***, respectively.*

We have

$$\begin{array}{ll}
D_{\mathcal{F}}(x_0) = 0, & D_{\mathcal{L}}(x_0) = 0, \\
D_{\mathcal{F}}(x_1) = 0, & D_{\mathcal{L}}(x_1) = x_0, \\
D_{\mathcal{F}}(x_2) = x_1, & D_{\mathcal{L}}(x_2) = 2x_1 \\
D_{\mathcal{F}}(x_3) = 2x_2, & D_{\mathcal{L}}(x_3) = 3\,x_2 - 3\,x_0, \\
D_{\mathcal{F}}(x_4) = 3\,x_3 - x_1, & D_{\mathcal{L}}(x_4) = 4\,x_3 - 4\,x_1, \\
D_{\mathcal{F}}(x_5) = 4\,x_4 - 2\,x_2, & D_{\mathcal{L}}(x_5) = 5\,x_4 - 5\,x_2 \\
D_{\mathcal{F}}(x_6) = 5\,x_5 - 3\,x_3 + x_1, & D_{\mathcal{L}}(x_6) = 6\,x_5 - 6\,x_3 + 6x_1.
\end{array}$$

We define the substitution homomorphisms $\varphi_{\mathcal{F}}, \varphi_{\mathcal{L}} : \mathbb{C}[x_0, x_1, \dots, x_n] \to \mathbb{C}[x]$ by $\varphi_{\mathcal{F}}(x_i) = F_i(x)$ and by $\varphi_{\mathcal{L}}(x_i) = L_i(x)$. Let

$$\ker \varphi_{\mathcal{F}} := \{P(x_0, x_1, ..., x_n) \mid \varphi_{\mathcal{F}}(P(x_0, x_1, ..., x_n)) = 0\},$$
$$\ker \varphi_{\mathcal{L}} := \{P(x_0, x_1, ..., x_n) \mid \varphi_{\mathcal{L}}(P(x_0, x_1, ..., x_n)) = 0\}.$$

From Theorem 1.1 it follows that $\varphi_{\mathcal{F}}(\ker \mathcal{D}_{\mathcal{F}}) \subset \ker \varphi_{\mathcal{F}}$ and $\varphi_{\mathcal{L}}(\ker \mathcal{D}_{\mathcal{L}}) \subset \ker \varphi_{\mathcal{L}}$. Note that $\varphi_{\mathcal{L}}(\ker \mathcal{D}_{\mathcal{L}}) \neq \ker \varphi_{\mathcal{L}}$. In fact, we have $\varphi_{\mathcal{F}}(x_n - (x_2 x_{n-1} + x_{n-2})) = F_n(x) - (F_2(x) F_{n-1}(x) + F_{n-2}(x)) = 0$ but $x_n - x_2 x_{n-1} - x_{n-2} \notin \ker \mathcal{D}_{\mathcal{F}}$.

2.2. **The kernel of the Fibonacci derivation.** It is obvious that this derivation is triangular and thus locally nilpotent. Thus, to find its kernel we may use Theorem 2.1.

Let us construct the Diximier map for the Fibonacci derivation. For this purpose, we first derive a closed-form expression for $D^k_{\mathcal{F}}(x_n)$.

We have

$$\begin{aligned} D^2_{\mathcal{F}}(x_n) &= D_{\mathcal{F}}\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) x_{n-1-2k} \right) \\ &= \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) \sum_{j=0}^{\left[\frac{n-2-2k}{2}\right]} (-1)^j (n-2-2k-2j) x_{n-2-2j} \\ &= \sum_{i=0}^{\left[\frac{n-2}{2}\right]} (-1)^i (i+1)(n-2-2i)(n-(i+1)) x_{n-2-2i}. \end{aligned}$$

Similarly we get

$$D^3_{\mathcal{F}}(x_n) = \sum_{i=0}^{\left[\frac{n-3}{2}\right]} (-1)^i \frac{(i+1)(i+2)}{2} (n-3-2i)(n-(i+1))(n-(i+2)) x_{n-3-2i}.$$

Induction gives

$$D^k_{\mathcal{F}}(x_n) = (k-1)! \sum_{i=0}^{\left[\frac{n-k}{2}\right]} (-1)^i (n-k-2i) \binom{i+k-1}{k-1} \binom{n-i-1}{k-1} x_{n-k-2i}.$$

Since $D_{\mathcal{F}}\left(-\frac{x_2}{x_1}\right) = -1$ we let $\lambda = -\frac{x_2}{x_1}$. Now we may find the Diximier map:

$$\begin{aligned} \sigma(x_n) &= \sum_{k=0}^{n-1} D^k_{\mathcal{F}}(x_n) \frac{\lambda^k}{k!} \\ &= x_n + \sum_{k=1}^{n-1} \frac{\lambda^k}{k} \sum_{i=0}^{\left[\frac{n-k}{2}\right]} (-1)^{k+i} (n-k-2i) \binom{i+k-1}{k-1} \binom{n-i-1}{k-1} x_{n-k-2i} \\ &= x_n + \sum_{k=1}^{n-3} \frac{\lambda^k}{k} \sum_{i=0}^{\left[\frac{n-k}{2}\right]} (-1)^i (n-k-2i) \binom{i+k-1}{k-1} \binom{n-i-1}{k-1} x_{n-k-2i} \\ &\quad + (n-1) x_2 \lambda^{n-2} + x_1 \lambda^{n-1}. \end{aligned}$$

Replacing λ by $-\dfrac{x_2}{x_1}$, we obtain, after simplifying:

$$x_1^{n-2}\sigma(x_n)$$
$$= x_n x_1^{n-2} + \sum_{k=1}^{n-3}\frac{1}{k}\sum_{i=0}^{\left[\frac{n-k}{2}\right]}(-1)^{k+i}(n-k-2i)\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}x_{n-k-2i}x_2^k x_1^{n-2-k}$$
$$+ (n-2)(-1)^{n-2}x_2^{n-1}.$$

The polynomials

$$C_n := x_n x_1^{n-2} + \sum_{k=1}^{n-3}\frac{1}{k}\sum_{i=0}^{\left[\frac{n-k}{2}\right]}(-1)^{k+i}(n-k-2i)\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}x_{n-k-2i}x_2^k x_1^{n-2-k}$$
$$+ (n-2)(-1)^{n-2}x_2^{n-1}, n > 2,$$

belong to the kernel $\ker \mathcal{D}_{\mathcal{F}}$. We call them *the Cayley elements* of the locally nilpotent derivation $\mathcal{D}_{\mathcal{F}}$. The first few Cayley elements are shown below:

$$C_3 = -{x_2}^2 + x_3x_1,$$
$$C_4 = 2\,{x_2}^3 - 3\,x_2x_3x_1 + {x_1}^2x_2 + x_4{x_1}^2,$$
$$C_5 = -3\,{x_2}^4 + 6\,{x_2}^2x_3x_1 - {x_1}^2{x_2}^2 - 4\,x_2x_4{x_1}^2 + x_5{x_1}^3,$$
$$C_6 = 4\,{x_2}^5 - 10\,{x_2}^3x_3x_1 - 2\,{x_1}^2{x_2}^3 + 10\,{x_2}^2x_4{x_1}^2 - 5\,x_2x_5{x_1}^3 + 3\,{x_1}^3x_2x_3 - {x_1}^4x_2 + x_6{x_1}^4.$$

Theorem 2.1 implies the following theorem.

Theorem 2.2.

$$\ker \mathcal{D}_{\mathcal{F}} = \mathbb{C}[x_0, x_1, C_3, C_4, \ldots, C_n][x_1^{-1}] \cap \mathbb{C}[x_0, x_1, \ldots, x_n].$$

Thus we obtain a description of the kernel of the Fibonacci derivation. In particular, $\varphi_F(C_n)$ is a constant and gives an identity for the Fibonacci polynomials. To find the explicit identity, we calculate $\varphi_F(C_n)$. We have

$$\varphi_F(C_n) = \varphi_F(C_n(x_0, x_1, \ldots, x_n)) = F_n(x)$$
$$+ \sum_{k=1}^{n-3}\frac{1}{k}\sum_{i=0}^{\left[\frac{n-k}{2}\right]}(-1)^{k+i}(n-k-2i)\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}F_{n-k-2i}(x)F_2(x)^k +$$
$$+ (n-2)(-1)^{n-2}F_2(x)^{n-1} = \text{const}.$$

Calculations for small n suggest the following conjecture:

Conjecture. $\varphi_F(C_n) = C_n(F_1(x), \ldots, F_n(x)) = \begin{cases} 0, & n \text{ even},\\ 1, & n \text{ odd}.\end{cases}$

2.3. **The kernel of the Lucas derivation.** Following the methods of the previous section, for the locally nilpotent derivation $D_{\mathcal{L}}$ we obtain:

$$D_{\mathcal{L}}^k(x_n) = n\,(k-1)!\sum_{i=0}^{\left[\frac{n-k}{2}\right]}(-1)^i\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}x_{n-k-2i}.$$

Since $D_{\mathcal{L}}\left(-\frac{x_1}{x_0}\right) = -1$ we put $\lambda = -\frac{x_1}{x_0}$. Then the Diximier map has the form:

$$\sigma(x_n) = \sum_{k=0}^{n} D_{\mathcal{L}}^k(x_n)\frac{\lambda^k}{k!}$$
$$= x_n + n\sum_{k=1}^{n-2}\frac{\lambda^k}{k}\sum_{n-k-2i>0}(-1)^{k+i}\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}x_{n-k-2i} + nx_1\lambda^{n-1} + x_0\lambda^n.$$

The corresponding Cayley elements are as follows:

$$C_n = x_n x_0^{n-1}$$
$$+ n\sum_{k=1}^{n-2}\frac{1}{k}\sum_{i=0}^{\left[\frac{n-k}{2}\right]}(-1)^{k+i}\binom{i+k-1}{k-1}\binom{n-i-1}{k-1}x_{n-k-2i}x_1^k x_0^{n-1-k} + (n-1)(-1)^{n-1}x_1^n, n > 2.$$

The first few Cayley polynomials for the Lucas derivation are shown below:

$$C_1 = x_0,$$
$$C_2 = x_2x_0 - {x_1}^2,$$
$$C_3 = 2\,{x_1}^3 + 3\,x_1{x_0}^2 - 3\,x_1x_2x_0 + x_3{x_0}^2,$$
$$C_4 = -3\,{x_1}^4 - 4\,{x_1}^2{x_0}^2 + 6\,{x_1}^2x_2x_0 - 4\,x_1x_3{x_0}^2 + x_4{x_0}^3,$$
$$C_5 = 4\,{x_1}^5 + 10\,{x_1}^2x_3{x_0}^2 - 5\,x_1{x_0}^4 - 5\,x_1x_4{x_0}^3 - 10\,{x_1}^3x_2x_0 + 5\,x_1x_2{x_0}^3 + x_5{x_0}^4.$$

Theorem 2.1 implies the following statement.

Theorem 2.3.

$$\ker \mathcal{D}_{\mathcal{L}} = \mathbb{C}[x_0, C_2, C_3, C_4, \ldots, C_n][x_0^{-1}] \cap \mathbb{C}[x_0, x_1, \ldots, x_n].$$

Thus, as in the previous section, $\varphi_L(C_n) = C_n(L_0(x), \ldots, L_n(x))$ is a constant.

Conjecture 2. $C_n(L_0(x), \ldots, L_n(x)) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$

3. Appel-Lucas and Appel-Fibonacci Intertwining Maps

3.1. **Appel-Lucas intertwining map.** Define a map ψ_{AL} by

$$\psi_{AL}(x_n) = x_n + \alpha_n^{(1)}x_{n-2} + \alpha_n^{(2)}x_{n-4} + \cdots + \alpha_n^{(i)}x_{n-2i} + \cdots + \alpha_n^{(\left[\frac{n-1}{2}\right])}x_{n-2\left[\frac{n-1}{2}\right]}.$$

Let us find a condition on ψ_{AL} to be an Appel-Lucas intertwining map. Let us prove the following statement.

Lemma 3.1. *The sequences* $\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(\left[\frac{n-1}{2}\right])}$ *satisfy the following system of recurrence equations:*

$$\begin{cases} (n-2)\alpha_n^{(1)} = n(\alpha_{n-1}^{(1)} + 1), \\ (n-4)\alpha_n^{(2)} = n(\alpha_{n-1}^{(2)} + \alpha_{n-1}^{(1)}), \\ (n-2i)\alpha_n^{(i)} = n(\alpha_{n-1}^{(i)} + \alpha_{n-1}^{(i-1)}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (n-2\left[\frac{n-1}{2}\right])\alpha_n^{(\left[\frac{n-1}{2}\right])} = n\left(\alpha_{n-1}^{(\left[\frac{n-1}{2}\right])} + \alpha_{n-1}^{(\left[\frac{n-1}{2}\right]-1)}\right), \end{cases}$$

with the initial conditions $\alpha_{2i}^{(i)} = 0$.

Proof. We have

$$D_{\mathcal{L}}(\psi_{AL}(x_n)) = n(x_{n-1} - x_{n-3} + x_{n-5} - x_{n-7} + \cdots) + \alpha_n^{(1)}(n-2)(x_{n-3} - x_{n-5} + \cdots) \\ + \alpha_n^{(2)}(n-4)(x_{n-5} - x_{n-7} + \ldots) + \cdots .$$

This gives

$$D_{\mathcal{L}}(\psi_{AL}(x_n)) = nx_{n-1} + x_{n-3}((n-2)\alpha_n^{(1)} - n) + x_{n-5}((n-4)\alpha_n^{(2)} - (n-2)\alpha_n^{(1)} + n) \\ + x_{n-2i-1}((n-2i)\alpha_n^{(i)} - (n-2(i-1))\alpha_n^{(i-1)} + (n-2(i-2))\alpha_n^{(i-2)} + \cdots) + \cdots .$$

On the other hand

$$D_{\mathcal{L}}(\psi_{AL}(x_n)) = \psi_{AL}(\mathcal{D}_{\mathcal{L}}(x_n)) = n\psi_{AL}(x_{n-1}) = n \sum_{n-1-2i>0} \alpha_{n-1}^{(i)} x_{n-1-2i}(-1)^i.$$

By equating the corresponding coefficients in the above two equations, we obtain the recurrence relations for the sequences $\alpha_n^{(i)}$:

$$(n-2)\alpha_n^{(1)} - n = n\alpha_{n-1}^{(1)}, \alpha_2^{(1)} = 0,$$
$$(n-4)\alpha_n^{(2)} - (n-2)\alpha_n^{(1)} + n = n\alpha_{n-1}^{(2)}, \alpha_4^{(1)} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sum_{n-2i>0} (n-2i)\alpha_n^{(i)}(-1)^i = n\alpha_{n-1}^{(i)}, \alpha_{2i}^{(i)} = 0.$$

Finally, the simplification yields

$$\begin{cases} (n-2)\alpha_n^{(1)} = n(\alpha_{n-1}^{(1)} + 1), \\ (n-4)\alpha_n^{(2)} = n(\alpha_{n-1}^{(2)} + \alpha_{n-1}^{(1)}), \\ (n-2i)\alpha_n^{(i)} = n(\alpha_{n-1}^{(i)} + \alpha_{n-1}^{(i-1)}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (n - 2\left[\frac{n-1}{2}\right])\alpha_n^{(\left[\frac{n-1}{2}\right])} = n\left(\alpha_{n-1}^{(\left[\frac{n-1}{2}\right])} + \alpha_{n-1}^{(\left[\frac{n-1}{2}\right]-1)}\right), \end{cases}$$

and $\alpha_{2i}^{(i)} = 0$. □

Let g_n be a fixed sequence and consider the auxiliary recurrence equation

$$(n-a)x_n = n(x_{n-1} + g_{n-1}), x_a = 0, n \geq a. \tag{3.1}$$

Then we have the following lemma.

Lemma 3.2. *Let*

$$x_n = n^{\underline{a}} \sum_{i=a}^{n-1} \frac{g_i}{i^{\underline{a}}},$$

where $n^{\underline{a}} := n(n-1)(n-2)\cdots(n-(a-1))$. *Then the sequence* x_n *is a solution of the recurrence equation (3.1).*

Proof. In fact, suppose that

$$x_n = n^{\underline{a}} \sum_{i=a}^{n-1} \frac{g_i}{i^{\underline{a}}}.$$

Then

$$\begin{aligned} n(x_{n-1}+g_{n-1}) &= n\left((n-1)^{\underline{a}}\sum_{i=a}^{n-2}\frac{g_i}{i^{\underline{a}}}+g_{n-1}\right) \\ &= n\left((n-1)^{\underline{a}}\sum_{i=a}^{n-2}\frac{g_i}{i^{\underline{a}}}+\frac{(n-1)^{\underline{a}}g_{n-1}}{(n-1)^{\underline{a}}}\right) \\ &= n(n-1)^{\underline{a}}\left(\sum_{i=a}^{n-2}\frac{g_i}{i^{\underline{a}}}+\frac{g_{n-1}}{(n-1)^{\underline{a}}}\right) \\ &= n(n-1)^{\underline{a}}\sum_{i=a}^{n-1}\frac{g_i}{i^{\underline{a}}}=(n-a)n^{\underline{a}}\sum_{i=a}^{n-1}\frac{g_i}{i^{\underline{a}}}=(n-a)x_n. \end{aligned}$$

□

By using the result we obtain

$$\alpha_n^{(1)}=n(n-2),$$

$$\alpha_n^{(2)}=\frac{1}{4}\,(n-1)\,n\,(3\,n-7)\,(n-4)=\frac{1}{2!}(n-4)\binom{n}{2}(3n-7),$$

$$\alpha_n^{(3)}=\frac{1}{3!}(n-6)\binom{n}{3}\left(19\,n^2-141\,n+254\right),$$

$$\alpha_n^{(4)}=\frac{1}{4!}(n-8)\binom{n}{4}\left(211\,n^3-3258\,n^2+16481\,n-27306\right),$$

$$\alpha_n^{(5)}=\frac{1}{5!}(n-10)\binom{n}{5}\left(3651\,n^4-96550\,n^3+946185\,n^2-4071950\,n+6492024\right).$$

To find the general solution of the system let us consider the sequence $\alpha_n^{(s)}$ in a basis of the falling powers. Let

$$\alpha_n^{(s)}=\beta_0^{(s)}n^{\underline{s}}+\beta_1^{(s)}n^{\underline{s+1}}+\cdots+\beta_s^{(s)}n^{\underline{2s}}.$$

It is easy to see that

$$(n-s)n^{\underline{s}}=n^{\overline{s+1}},n(n-1)^{\underline{s}}=n^{\overline{s+1}}.$$

Then

$$\begin{aligned} (n-2s)\alpha_n^{(s)} &= (n-2s)(\beta_0^{(s)}n^{\underline{s}}+\beta_1^{(s)}n^{\underline{s+1}}+\cdots+\beta_s^{(s)}n^{\underline{2s}}) \\ &= ((n-s)-s)\beta_0^{(s)}n^{\overline{s}}+((n-(s+1))-(s-1)))\beta_1^{(s)}n^{\underline{s+1}}+\cdots+(n-2s)\beta_s^{(s)}n^{\underline{2s}} \\ &= \sum_{i=0}^{s}\beta_i^{(s)}n^{\overline{s+i+1}}-\sum_{i=0}^{s-1}(s-i)\beta_i^{(s)}n^{\overline{s+i}}. \end{aligned}$$

On the other hand

$$\begin{aligned} n(\alpha_{n-1}^{(s)}+\alpha_{n-1}^{(s-1)}) &= \sum_{i=0}^{s}n\beta_i^{(s)}(n-1)^{\overline{s+i}}+\sum_{i=0}^{s-1}n\beta_i^{(s-1)}(n-1)^{\overline{s-1+i}} \\ &= \sum_{i=0}^{s}\beta_i^{(s)}n^{\overline{s+i+1}}+\sum_{i=0}^{s-1}\beta_i^{(s-1)}n^{\overline{s+i}}. \end{aligned}$$

By equating the corresponding coefficients we obtain

$$\beta_i^{(s)} = \frac{\beta_i^{(s-1)}}{i-s}, i = 0, \dots, s-1.$$

The coefficient $\beta_s^{(s)}$ is found from the initial condition $a_{2s}^{(s)} = 0$. We have

$$\sum_{i=0}^{s-1} \frac{\beta_i^{(s-1)}}{i-s} (2s)^{\underline{s+i}} + \beta_s^{(s)} (2s)! = 0.$$

It follows that

$$\beta_s^{(s)} = \frac{1}{(2s)!} \sum_{i=0}^{s-1} \frac{\beta_i^{(s-1)}}{s-i} (2s)^{\underline{s+i}} = \sum_{i=0}^{s-1} \frac{\beta_i^{(s-1)}}{(s-i)(s-i)!} = -\sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i)!}.$$

Therefore, we get the following recurrence relations for the sequences $\beta_n^{(s)}$:

$$\begin{aligned}
\beta_0^{(s)} &= -\frac{\beta_0^{(s-1)}}{s}, \\
\beta_1^{(s)} &= -\frac{\beta_1^{(s-1)}}{s-1}, \\
\beta_2^{(s)} &= -\frac{\beta_1^{(s-1)}}{s-2}, \\
&\dots \\
\beta_{s-1}^{(s)} &= -\beta_1^{(s-1)}, \\
\beta_s^{(s)} &:= b_s = -\sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i)!}.
\end{aligned}$$

Thus,

$$\beta_i^{(s)} = \frac{(-1)^{s-i}}{(s-i)!} b_i, \text{ for } i = 0, \dots, s-1.$$

Therefore, we get the following recurrence relation for the sequence b_s :

$$\sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!^2} b_n = 0, n > 0.$$

Recall the definition of the Bessel function $J_\alpha(x)$:

$$J_\alpha(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i\Gamma(i+\alpha+1)} \left(\frac{z}{2}\right)^{2i+\alpha}.$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} z^n = J_0(\sqrt{4z}),$$

and the ordinary generating function

$$G(b_n, z) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, the above recurrence relation implies that

$$J_0(\sqrt{4z})G(b_n, z) = 1.$$

We have

$$G(b_n, z) = \frac{1}{J_0(\sqrt{4z})} = 1 + z + \frac{3}{4}z^2 + \frac{19}{36}z^3 + \frac{211}{576}z^4 + \frac{1217}{4800}z^5 + \frac{30307}{172800}z^6 + \cdots .$$

Thus we prove the following statement.

Theorem 3.1. *Define the linear map* ψ_{AL} *by*

$$\psi_{AL}(x_n) = x_n + \alpha_n^{(1)}x_{n-2} + \alpha_n^{(2)}x_{n-4} + \cdots + \alpha_n^{(i)}x_{n-2i} + \cdots + \alpha_n^{([\frac{n-1}{2}])}x_{n-2[\frac{n-1}{2}]},$$

where

$$\alpha_n^{(s)} = \frac{(-1)^s}{s!}b_0 n^{\underline{s}} + \cdots + \frac{(-1)^{s-i}}{(s-i)!}\, b_i n^{\underline{s+i}} + \cdots + b_s n^{\underline{2s}},$$

and the generating function for $b_0, b_1, \ldots, b_n, \ldots$ *is*

$$\sum_{i=0}^{\infty} b_i z^i = J_0^{-1}(\sqrt{4z}).$$

Then ψ_{AL} *is an Appel-Lucas intertwining map.*

3.2. **Appel-Fibonacci intertwining map.** Define a linear map ψ_{AF} by

$$\psi_{AF}(x_n) = \alpha_n^{(0)}x_{n+1} + \alpha_n^{(1)}x_{n-1} + \alpha_n^{(2)}x_{n-3} + \cdots + \alpha_n^{(i)}x_{n+1-2i} + \cdots + \alpha_n^{([\frac{n-1}{2}])}x_{n+1-2[\frac{n-1}{2}]}.$$

Let us prove the following statement.

Lemma 3.3. *If* ψ_{AF} *is an Appel-Fibonacci intertwining map then the coefficients* $\alpha_n^{(1)}$, $\alpha_n^{(2)}, \ldots, \alpha_n^{([\frac{n-1}{2}])}$ *satisfy the following system of recurrence equations:*

$$\begin{cases} \alpha_n^{(0)} = 1, \\ \alpha_n^{(1)} = n\left(\dfrac{\alpha_{n-1}^{(1)}}{n-2} + \dfrac{\alpha_{n-1}^{(0)}}{n}\right), \\ \alpha_n^{(2)} = n\left(\dfrac{\alpha_{n-1}^{(2)}}{n-4} + \dfrac{\alpha_{n-1}^{(1)}}{n-2}\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n^{(s)} = n\left(\dfrac{\alpha_{n-1}^{(s)}}{n-2s} + \dfrac{\alpha_{n-1}^{(s-1)}}{n-2(s-1)}\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

with initial conditions $\alpha_{2i}^{(i)} = 0$.

Proof. We have

$$D_{\mathcal{F}}(\psi_{AF}(x_n)) = \alpha_n^{(0)}(nx_n - (n-2)x_{n-2} + (n-4)x_{n-4} - (n-6)x_{n-6} + \cdots)$$
$$+ \alpha_n^{(1)}((n-2)x_{n-2} - (n-4)x_{n-4} + \cdots) + \alpha_n^{(2)}((n-4)x_{n-4} - (n-6)x_{n-6} + \cdots) + \cdots .$$

It implies

$$D_{\mathcal{F}}(\psi_{AF}(x_n)) = n\alpha_n^{(0)}x_n + (n-2)x_{n-2}(\alpha_n^{(1)} - \alpha_n^{(0)}) + (n-4)x_{n-4}(\alpha_n^{(2)} - \alpha_n^{(1)} + \alpha_n^{(0)}) + x_{n-2i}(n-2i)(\alpha_n^{(i)} - \alpha_n^{(i-1)} + \alpha_n^{(i-2)} + \cdots + (-1)^i\alpha_n^{(0)}) + \cdots .$$

On the other hand

$$D_{\mathcal{F}}(\psi_{AF}(x_n)) = \psi_{AF}(\mathcal{D}_{\mathcal{F}}(x_n)) = n\psi_{AF}(x_{n-1}) = n\sum_{i=0}^{n-1}\alpha_{n-1}^{(i)}x_i.$$

Thus we find such recurrence relations for $\alpha_n^{(i)}$:

$$\begin{aligned}
&n\alpha_n^{(0)} = n\alpha_{n-1}^{(0)},\\
&(n-2)(\alpha_n^{(1)} - \alpha_n^{(0)}) = n\alpha_{n-1}^{(1)}, \alpha_2^{(1)} = 0,\\
&(n-4)(\alpha_n^{(2)} - \alpha_n^{(1)} + \alpha_n^{(0)}) = n\alpha_{n-1}^{(2)}, \alpha_4^{(1)} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(n-2i)\sum_{n-2i>0}\alpha_n^{(i)}(-1)^i = n\alpha_{n-1}^{(i)}, \alpha_{2i}^{(i)} = 0.
\end{aligned}$$

Finally, the simplification yields

$$\begin{cases}
\alpha_n^{(0)} = 1,\\
\alpha_n^{(1)} = n\left(\dfrac{\alpha_{n-1}^{(1)}}{n-2} + \dfrac{\alpha_{n-1}^{(0)}}{n}\right),\\
\alpha_n^{(2)} = n\left(\dfrac{\alpha_{n-1}^{(2)}}{n-4} + \dfrac{\alpha_{n-1}^{(1)}}{n-2}\right),\\
\ldots\\
\alpha_n^{(s)} = n\left(\dfrac{\alpha_{n-1}^{(2)}}{n-2s} + \dfrac{\alpha_{n-1}^{(1)}}{n-2(s-1)}\right).\\
\ldots.
\end{cases}$$

□

Let g_n be some fixed sequence and consider the auxiliary recurrence equation

$$x_n = n\left(\frac{x_{n-1}}{n-s} + \frac{g_{n-1}}{n-(s-2)}\right), x_s = 0.$$

Then we have the following lemma.

Lemma 3.4. *The solution of this auxiliary recurrence equation is*

$$x_n = n^{\underline{s}}\sum_{i=s}^{n-1}\frac{g_i}{i^{\underline{s-1}}\ (i-(s-3))}.$$

Proof. Using the relations,

$$\frac{n(n-1)^{\underline{s}}}{n-1} = n^{\underline{s}}, n(n-1)^{\underline{s-1}} = n^{\underline{s}}.$$

we have

$$n\left(\frac{x_{n-1}}{n-s}+\frac{g_{n-1}}{n-(s-2)}\right)=n\left(\frac{(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}+\frac{g_{n-1}}{n-(s-2)}\right)$$
$$=n\left(\frac{(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}+\frac{(n-1)^{\underline{s-1}}g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))}\right)$$
$$=\frac{n(n-1)^{\underline{s}}}{n-s}\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}+\frac{n(n-1)^{\underline{s-1}}g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))}$$
$$=n^{\underline{s}}\left(\sum_{i=s}^{n-2}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}+\frac{g_{n-1}}{((n-1)^{\underline{s-1}})((n-1)-(s-3))}\right)$$
$$=n^{\underline{s}}\sum_{i=s}^{n-1}\frac{g_i}{i^{\underline{s-1}}\,(i-(s-3))}.$$

□

By using the result of the Lemma, we obtain

$$\alpha_n^{(0)}=1,$$
$$\alpha_n^{(1)}=\frac{1}{2}(n-1)(n-2),$$
$$\alpha_n^{(2)}=\frac{1}{6}\,(n-4)\,(n-3)\,(n-2)\,n,$$
$$\alpha_n^{(3)}=\frac{1}{144}\,(n-1)\,n\,(n-4)\,(n-5)\,(7\,n-17)\,(n-6)\,,$$
$$\alpha_n^{(4)}=\frac{1}{2880}\,(n-8)\left(39\,n^2-296\,n+545\right)(n-7)\,(n-6)\,(n-2)\,(n-1)\,n.$$

To find a solution of the system let us consider the unknown sequence $\alpha_n^{(s)}$ in a basis of the falling powers. Let

$$\alpha_n^{(s)}=(n-(2s-1))\left(\beta_0^{(s)}n^{\underline{s-1}}+\beta_1^{(s)}n^{\underline{s}}+\cdots+\beta_s^{(s)}n^{\underline{2s-1}}\right)=(n-(2s-1))\sum_{i=0}^{s}\beta_i^{(s)}n^{\underline{s-1+i}}.$$

Then

$$\alpha_n^{(s)}=(n-(2s-1))\sum_{i=0}^{s}\beta_i^{(s)}n^{\underline{s-1+i}}=\sum_{i=0}^{s-1}\beta_i^{(s)}(n-(s+i-1)-(s-i))n^{\underline{s+i}}$$
$$=\sum_{i=0}^{s}\beta_i^{(s)}(n-(s+i-1))n^{\underline{s+i-1}}-\sum_{i=0}^{s}\beta_i^{(s)}(s-i)n^{\underline{s-1+i}}=\sum_{i=0}^{s}\beta_i^{(s)}n^{\underline{s+i}}-\sum_{i=0}^{s-1}(s-i)\beta_i^{(s)}n^{\underline{s-1+i}}.$$

On the other hand

$$\alpha_{n-1}^{(s)}=(n-2s)\sum_{i=0}^{s}\beta_i^{(s)}(n-1)^{\underline{s-1+i}},$$
$$\alpha_{n-1}^{(s-1)}=(n-(2s-2))\sum_{i=0}^{s-1}\beta_i^{(s-1)}(n-1)^{\underline{s+i-2}}.$$

Then

$$\begin{aligned}\alpha_n^{(s)} &= n\left(\frac{\alpha_{n-1}^{(s)}}{n-2s} + \frac{\alpha_{n-1}^{(s-1)}}{n-(2s-2)}\right)\\ &= \sum_{i=0}^{s} n\beta_i^{(s)}(n-1)^{\underline{s-1+i}} + \sum_{i=0}^{s-1} n\beta_i^{(s-1)}(n-1)^{\underline{s-2+i}}\\ &= \sum_{i=0}^{s} \beta_i^{(s)} n^{\underline{s+i}} + \sum_{i=0}^{s-1} \beta_i^{(s-1)} n^{\underline{s-1+i}}.\end{aligned}$$

Thus,

$$\sum_{i=0}^{s-1} \beta_i^{(s-1)} n^{\underline{s-1+i}} = -\sum_{i=0}^{s-1} (s-i)\beta_i^{(s)} n^{\underline{s-1+i}}.$$

By equating the corresponding coefficients of the $n^{\underline{i}}$, we get

$$\beta_i^{(s)} = -\frac{\beta_i^{(s-1)}}{s-i}, i = 0, \ldots, s-1.$$

The coefficient $\beta_s^{(s)}$ is determined from the initial condition $\alpha_{2s}^{(s)} = 0$. We have

$$\alpha_{2s}^{(s)} = \sum_{i=0}^{s} \beta_i^{(s)} (2s)^{\underline{s-1+i}} = \sum_{i=0}^{s-1} \beta_i^{(s)} (2s)^{\underline{s-1+i}} + \beta_s^{(s)} (2s)^{\underline{2s-1}} = 0.$$

Taking into account $(2s)^{\underline{2s-1}} = 2s(2s-1)\ldots 2 = (2s)!$ we have

$$\beta_s^{(s)} = -\frac{1}{(2s)!}\sum_{i=0}^{s-1} \beta_i^{(s)} (2s)^{\underline{s-1+i}} = -\sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i+1)!}.$$

Thus we get the following relations for $\beta_n^{(s)}$:

$$\begin{aligned}\beta_0^{(s)} &= -\frac{\beta_0^{(s-1)}}{s},\\ \beta_1^{(s)} &= -\frac{\beta_1^{(s-1)}}{s-1},\\ \beta_2^{(s)} &= -\frac{\beta_1^{(s-1)}}{s-2},\\ &\ldots\\ \beta_{s-1}^{(s)} &= -\beta_1^{(s-1)},\\ \beta_s^{(s)} &:= b_s = -\sum_{i=0}^{s-1} \frac{\beta_i^{(s)}}{(s-i+1)!}.\end{aligned}$$

It yields

$$\beta_i^{(s)} = \frac{(-1)^{s-i}}{(s-i)!} b_i, \text{ for } i = 0, \ldots, s-1,$$

and

$$b_s = -\frac{(-1)^s}{((s+1)!)^2} - \sum_{i=1}^{s-1} \frac{(-1)^{s-i}}{((s-i)(s-i+1)!)} b_i.$$

Therefore we get the following recurrence relation for the sequence b_s:

$$\sum_{i=0}^{s} \frac{(-1)^{n-i}}{(s-i)!(s-i+1)!} b_n = 0.$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} z^n = \frac{1}{\sqrt{z}} J_1(\sqrt{4z}\,)$$

and write

$$G(b_n, z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then taking into account the recurrence relation we find that

$$G(b_n, z) \frac{J_1(\sqrt{4z}\,)}{\sqrt{z}} = 1.$$

Below are a few initial terms

$$G(b_n, z) = \frac{\sqrt{z}}{J_1(\sqrt{4z}\,)} = 1 + \frac{1}{2}\, z + \frac{1}{6}\, z^2 + \frac{7}{144}\, z^3 + \frac{13}{960}\, z^4 + \frac{107}{28800}\, z^5 + \frac{409}{403200}\, z^6 + \cdots .$$

Thus we have the following statement.

Theorem 3.2. *Define the linear map ψ_{AF} by*

$$\psi_{AL}(x_n) = x_{n+1} + \alpha_n^{(1)} x_{n-1} + \alpha_n^{(2)} x_{n-3} + \cdots + \alpha_n^{(i)} x_{n+1-2i} + \cdots + \alpha_n^{([\frac{n-1}{2}])} x_{n+1-2[\frac{n-1}{2}]},$$

where

$$\alpha_n^{(s)} = (n-2s+1)\left(\frac{(-1)^s}{s!} b_0 n^{\underline{s-1}} + \cdots + \frac{(-1)^{s-i}}{(s-i)!}\, b_i n^{\underline{s+i}} + \cdots + b_s n^{\underline{2s-1}} \right),$$

and the ordinary generating function for $b_0, b_1, \ldots, b_n, \ldots$ is as follows:

$$\sum_{i=0}^{\infty} b_i z^i = \frac{\sqrt{z}}{J_1(\sqrt{4z}\,)}.$$

Then ψ_{AF} is an Appel-Fibonacci intertwining map.

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