FIBONACCI NUMBERS AND IDENTITIES

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Abstract. By investigating functions satisfying a recurrence relation, we give alternative proofs for various identities involving Fibonacci numbers and Lucas numbers. Then we make certain well-known identities *visible* via a certain trivalent graph (an infinite graph with all vertices of degree 3) associated with the recurrence relation.

1. Introduction

A function $x(n)$ defined for nonnegative integers is called an $\mathcal{F}$-function if $x(n)$ satisfies the recurrence relation

$$x(n+3) = 2(x(n+2) + x(n+1)) - x(n). \tag{1.1}$$

One easily sees that the following are $\mathcal{F}$-functions:

$$x(n) = (-1)^n,\ x(n) = F_n^2,\ x(n) = F_{n+r}F_n,\ x(n) = F_{2n}, \tag{1.2}$$

$$x(n) = L_n^2,\ x(n) = L_{n+r}L_n,\ x(n) = L_{2n},\ x(n) = F_nL_{n+r}. \tag{1.3}$$

Here r is an integer and F_n, L_n are the nth Fibonacci and Lucas numbers, respectively. Note that the sum and difference of $\mathcal{F}$-functions are $\mathcal{F}$-functions. A search of the literature shows that a great many identities involve only $\mathcal{F}$-functions. For instance, all the terms in Cassini's Identity $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ are $\mathcal{F}$-functions. For convenience, we shall call such an identity an $\mathcal{F}$-identity. Identities with other recurrence relations are less frequent. The main purpose of this article is to avoid the intricate case-by-case analysis and obtain a unified proof of the $\mathcal{F}$-identities. Since these identities involve only $\mathcal{F}$-functions, our proof will only make use of (1.1) and the facts that

(i) (1.2) and (1.3) are $\mathcal{F}$-functions,
(ii) $F_{n+2} = F_{n+1} + F_n$, $F_{-m} = (-1)^{m+1}F_m$, $L_{n+2} = L_{n+1} + L_n$, $L_{-m} = (-1)^m L_m$.

Note that our proof can be easily applied to all $\mathcal{F}$-identities. Identities involving other recurrence relations such as

(iii) $A(n+2) = -A(n+1) + A(n)$,
(iv) $A(n+3) = -2A(n+2) + 2A(n+1) + A(n)$

will be discussed in Section 6.

The rest of the article is organized as follows. In Section 2 we give some basic properties of $\mathcal{F}$-functions. Section 3 gives alternative proofs of the well-known Catalan's Identity and Melham's Identity. Section 4 lists a few more identities (including d'Ocagne's, Tagiure's and Gelin-Cesàro Identities) that can be proved by applying our technique presented in Section 3. They are the $\mathcal{F}$-identities involving functions we listed in (1.2) and (1.3). In other words, they use functions in (1.2) and (1.3) as building blocks (see Lemma 2.1). Since the product of $\mathcal{F}$-functions are not necessarily $\mathcal{F}$-functions, our idea cannot be applied to all identities. Section 5 is devoted to the possible visualization of identities via the recurrence relation (1.1).

After all, there is nothing to prove if one cannot *see* the identities in the first place. The last section gives a very brief discussion of identities that involve other recurrence relations.

2. Basic Properties About $\mathcal{F}$-Functions

Lemma 2.1. *The functions defined in* (1.2) *and* (1.3) *are $\mathcal{F}$-functions. In addition, let $A(n)$ and $B(n)$ be $\mathcal{F}$-functions and let r_0 be a fixed integer. Then $X(n) = A(n+r_0)$, $Y(n) = r_0A(n)$, and $A(n) \pm B(n)$ are $\mathcal{F}$-functions.*

Proof. To show $x(n)$ is an $\mathcal{F}$-function, it suffices to show that $x(n+3) = 2(x(n+2) + x(n+1)) - x(n)$. For example, the following shows that F_n^2 is an $\mathcal{F}$-function.

$$F_{n+3}^2 = (F_{n+2} + F_{n+1})^2 = F_{n+2}^2 + F_{n+1}^2 + 2F_{n+2}F_{n+1} = F_{n+2}^2 + F_{n+1}^2 + F_{n+2}(F_{n+2} - F_n)$$
$$+ (F_{n+1} + F_n)F_{n+1} = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n(F_{n+2} - F_{n+1}) = 2(F_{n+2}^2 + F_{n+1}^2) - F_n^2.$$

The proofs for the other functions in the lemma are similar. □

Lemma 2.2. *Let $A(n)$ and $B(n)$ be $\mathcal{F}$-functions. Then $A(n) = B(n)$ if and only if $A(k) = B(k)$ for $k = 0, 1$ and 2.*

Proof. Since $A(n)$ and $B(n)$ satisfy recurrence relation (1.1), $A(n) = B(n)$ if and only if they satisfy the same initial conditions. □

Example 2.3. *By Lemma 2.1, F_{n+3}^2, $F_{n+3}F_{n+4}$, L_{n+3}^2 and $F_{n+3}L_{n+3}$ are $\mathcal{F}$-functions.*

The following lemma is straightforward and will be used in Sections 3 and 6.

Lemma 2.4. *Let $A(n)$ and $B(n)$ be functions defined for nonnegative integers. Suppose that both $A(n)$ and $B(n)$ satisfy either*

(i) $x(n+2) = -x(n+1) + x(n)$, *or*
(ii) $x(n+3) = -2x(n+2) + 2x(n+1) + x(n)$.

Then $A(n) = B(n)$ if and only of $A(k) = B(k)$ for $k = 0, 1$ and 2.

2.1. **Discussion.** Let $\{x_n\}$ be a sequence that satisfies the recurrence relation $x_{n+2} = x_{n+1} + x_n$. Then $A(n) = x_{2n}$, $B(n) = x_nx_{n+r}$, and $C(n) = x_n^2$ are $\mathcal{F}$-functions.

3. An Alternative Proof for Catalan's Identities

In [4], Howard studied generalized Fibonacci sequences and proved that Catalan's Identity is equivalent to an identity discovered and proved by Melham [7] (see Section 3.1 of the present paper). In the following we give our alternative proof which uses only Lemmas 2.1 and 2.2.

Lemma 3.1. *Let r be an integer. Then $4(-1)^{3-r} + F_{r+3}F_{r-3} - F_r^2 = 0$.*

Proof. We assume that $r \geq 0$. The case $r \leq 0$ can be dealt with similarly. Let $A(r) = 4(-1)^{3-r} + F_{r+3}F_{r-3} - F_r^2$. By Lemma 2.1, $A(r)$ is an $\mathcal{F}$-function. By Lemma 2.2, $A(r) = 0$ for all r. This completes the proof of the lemma. □

Remark. Any $\mathcal{F}$-identity in one integer variable can be proved by applying the proof technique of Lemma 3.1. Cassini's Identity is an example.

Theorem 3.2. *(Catalan's Identity). Let r and n be integers. Then $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$.*

Proof. Recall first that $F_{-m} = (-1)^{m+1}F_m$. As a consequence, we may assume without loss of generality that $n, r \geq 0$. As a matter of fact, we may assume that $n \geq 1$ as the case $n = 0$ is trivial. Let $A(n) = F_n^2 - F_{n+r}F_{n-r}$, $B(n) = (-1)^{n-r}F_r^2$. By Lemma 2.1, both $A(n)$ and $B(n)$ are $\mathcal{F}$-functions (in n). Note that

$$A(1) = 1 - F_{1+r}F_{1-r},\ A(2) = 1 - F_{2+r}F_{2-r},\ A(3) = 4 - F_{3+r}F_{3-r}, \tag{3.1}$$

and

$$B(1) = (-1)^{1-r}F_r^2,\ B(2) = (-1)^{2-r}F_r^2,\ B(3) = (-1)^{3-r}F_r^2. \tag{3.2}$$

If $r \leq 3$, it follows that $A(n) = B(n)$ for $n = 1, 2, 3$. By Lemma 2.2, we have $A(n) = B(n)$. Hence, the theorem is proved in the case $r \leq 3$. We shall therefore assume that $r \geq 4$. Recall that $F_{-m} = (-1)^{m+1}F_m$. This allows us to rewrite $A(3)$ as $A(3) = 4 + (-1)^{3-r}F_{r+3}F_{r-3}$. Hence,

$$A(3) - B(3) = 4 + (-1)^{3-r}F_{r+3}F_{r-3} - (-1)^{3-r}F_r^2. \tag{3.3}$$

By Lemma 3.1, $A(3) = B(3)$. One can show similarly that $A(1) = B(1)$ and $A(2) = B(2)$. Applying Lemma 2.2, we have $A(n) = B(n)$ for all n. This completes the proof of the theorem. □

Remark. Any $\mathcal{F}$-identity in two integer variables can be proved by applying the proof technique of Theorem 3.2.

3.1. **Melham's Identity.** In [7], Melham proved, among some very general results, the identity $F_{n+r+1}^2 + F_{n-r}^2 = F_{2r+1}F_{2n+1}$. We shall give an alternative proof here. Denote by $A(n)$ and $B(n)$ the left- and right-hand side of the identity. By Lemma 2.1, $A(n)$ and $B(n)$ are $\mathcal{F}$-functions (in n). The cases $n = 0, 1$ and 2 are given by

$$F_{r+1}^2 + F_{-r}^2 = F_{2r+1}F_1,\ F_{r+2}^2 + F_{1-r}^2 = F_{2r+1}F_3,\ F_{r+3}^2 + F_{2-r}^2 = F_{2r+1}F_5. \tag{3.4}$$

By Lemma 2.1, the functions in (3.4) are $\mathcal{F}$-functions (in r) and the identities can be verified by applying Lemma 2.2. Consequently, we have $A(0) = B(0)$, $A(1) = B(1)$ and $A(2) = B(2)$. By Lemma 2.2, we have $A(n) = B(n)$ for all n.

3.2. **Discussion.** Our method can be generalized to functions such as $x(n) = F_n^3$ and $y(n) = F_{3n}$ which satisfy the recurrence relation (see Appendix B)

$$x(n+4) = 3x(n+3) + 6x(n+2) - 3x(n+1) - x(n). \tag{3.5}$$

4. More Identities

The purpose of this section is to list a few identities we found in the literature that can be proved by applying Lemmas 2.1 and 2.2.

4.1. **d'Ocagne's Identity.** The proof we presented in Section 3 can be applied to all $\mathcal{F}$-identities. A search of the literature reveals that there are many such identities. However, as the identities may be described in different manners, it is important to get equivalent forms of the identities. Take d'Ocagne's Identity for example. This identity is given as (see [9])

$$F_mF_{n+1} - F_nF_{m+1} = (-1)^nF_{m-n}. \tag{4.1}$$

A first look at the left- and right-hand sides does not reveal the fact that they are $\mathcal{F}$-function. However, one has the following. Let $r = m - n$. Then (4.1) can be rewritten as

$$F_{n+1}F_{n+r} - F_nF_{n+r+1} = (-1)^nF_r, \tag{4.2}$$

where both the left- and right-hand sides in (4.2) are $\mathcal{F}$-functions in terms of n. One may now apply the proof technique of Theorem 3.2 to obtain a proof of (4.2). Since the proof is similar to that of Theorem 3.2, we will not include it here.

4.2. **Some more identities.** A search of the literature reveals that there are many identities that can be verified by Lemmas 2.1 and 2.2 (for instance, out of the 44 identities given in Long [5], 35 of them involve $\mathcal{F}$-functions, see also [2] and [3]). We shall list a few here which we obtained mainly from [9] (($c1$) − −($c6$), ($c8$), ($c9$), ($d1$)).

($c1$)	$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_aF_b$	:	$F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2$	($d1$)
($c2$)	$L_n^2 - 5F_n^2 = 4(-1)^n$	:	$F_{2n+1} + (-1)^n = F_{n-1}F_{n+1} + F_{n+1}^2$	($d2$)
($c3$)	$F_mF_n = \frac{1}{5}(L_{m+n} - (-1)^n L_{m-n})$	:	$\sum_{i=1}^n L_i^2 = L_nL_{n+1} + 2$	($d3$)
($c4$)	$F_n^2 = \frac{1}{5}(L_{2n} - 2(-1)^n)$	:	$\sum_{i=1}^n F_i^2 = F_nF_{n+1}$	($d4$)
($c5$)	$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$	:	$F_nF_{n+3} = F_{n+1}F_{n+2} + (-1)^{n-1}$	($d5$)
($c6$)	$F_{m+n} = \frac{1}{2}(F_mL_n + L_mF_n)$	:	$F_n^2 - F_{n-1}^2 = F_nF_{n-1} + (-1)^{n-1}$	($d6$)
($c7$)	$L_{n+k+1}^2 + L_{n-k}^2 = 5L_{2k+1}L_{2n+1}$	:	$L_{2n+1} - F_{n+1}^2 - A = (-1)^{n-1}$	($d7$)
($c8$)	$F_n^2 + (-1)^{n+r-1}F_r^2 = F_{n-r}F_{n+r}$	:	$L_{n-1}^2 - F_{n-4}F_n - F_nF_{n+1} = F_{n-2}^2$	($d8$)
($c9$)	$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$	:	$F_{2n+1} = F_{n+3}F_n - F_{n+1}F_{n-1}$	($d9$)

where $A = (L_n^2 - F_{n-3}F_{n+1}) + F_{2n-2}$ and L_n is the nth Lucas number. Identities ($d7$) and ($d8$) are less standard. We decided to include them in the table as they are *visible* via a certain trivalent graph (see Section 5).

Proof. We first note that the functions in ($c9$) are not $\mathcal{F}$-functions but the identity can be proved by applying two $\mathcal{F}$-identities $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $F_n^2 - F_{n+2}F_{n-2} = (-1)^{n-2}$. In ($c3$), ($c5$), and ($c6$), one needs to rewrite the expressions to see that the functions are $\mathcal{F}$-functions. One may now apply Lemmas 2.1 and 2.2 and our proof technique presented in Theorem 3.2 to verify these identities. □

Remark. As identities may be described differently, the technique of rewriting identities into equivalent forms is crucial (see (4.1) and (4.2)).

4.3. **Discussion.** Note that in our proof, we do not use any existing identities such as Binet's Formula or any identities listed in [9] except for (1.1), (1.2), and (1.3), which is what we promised in our introduction. Note also that one has to apply Lemma 2.2 three times to prove identity ($c1$), known as Tagiuri's Identity.

5. How Far Can (1.1) Go?

We have demonstrated that the recurrence relation (1.1) can be used to verify various identities. In this section, we will present a trivalent graph (see the graph given in Appendix A) which is closely related to (1.1) that enables us to *visualize* identities in the following.

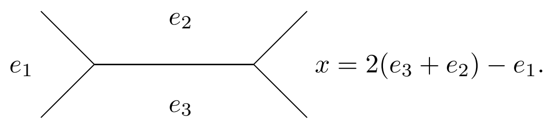
Let e_3, e_2, and e_1 be arbitrary vectors placed in the following trivalent graph and let e_4 be the vector given by

$$e_4 = 2(e_3 + e_2) - e_1. \tag{5.1}$$

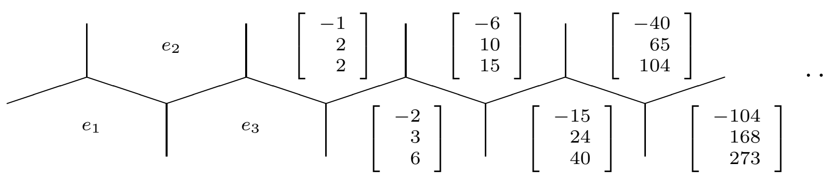
Such a vector e_4 is said to be $\mathcal{F}$-generated by e_3, e_2, and e_1 (in this order).

Note that (5.1) can be viewed as a generalization (in the form of vectors) of the recurrence relation (1.1). We may construct an infinite sequence of vectors given as follows.

$$e_1, e_2, e_3, e_4 = 2(e_3 + e_2) - e_1, \ldots, e_{n+1} = 2(e_n + e_{n-1}) - e_{n-2}, \ldots. \tag{5.2}$$



We denote the above sequence by $F(e_1, e_2, e_3)$. In the case $\{e_1, e_2, e_3\}$ is the canonical basis of 3-dimensional Euclidean space, the first nine vectors are given as follows. Consider the triples $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, $e_4 = (-1,2,2), e_5 = (-2,3,6), \ldots$. Note that $e_2, e_4, e_6, \ldots, e_{2n}$ make-up the top half of the graph and $e_1, e_3, e_5, \ldots, e_{2n+1}$ make-up the bottom half of the graph.



One immediately sees the following:

(a) The entries of the vectors $e_n = (a, b, c)$ are products of two Fibonacci numbers. To be more precise, for the first nine terms, the vectors take the form

$$(-F_nF_{n+1}, F_nF_{n+2}, F_{n+1}F_{n+2}). \tag{5.3}$$

(b) The norm of $e_n = (a, b, c)$ is the sum $a + b + c$, where the norm $N(x_1, x_2, x_3)$ is defined to be $(x_1^2 + x_2^2 + x_3^2)^{1/2}$ (see Appendix A).

(c) The absolute value of the entries of $e_{2n} - e_{2n-2}$ (the top half of the trivalent graph) and $e_{2n+1} - e_{2n-1}$ (the bottom half of the trivalent graph) are Fibonacci numbers. For instance, $(-40, 65, 104) - (-6, 10, 15) = (-F_9, F_{10}, F_{11})$. This allows us to write each entry of the vectors as a sum of Fibonacci numbers.

(a) and (b) of the above imply that for $n \le 6$

$$(F_nF_{n+1})^2 + (F_nF_{n+2})^2 + (F_{n+1}F_{n+2})^2 = (-F_nF_{n+1} + F_nF_{n+2} + F_{n+1}F_{n+2})^2, \tag{5.4}$$

which leads us to (i) of the following lemma. Note that $-F_nF_{n+1} + F_nF_{n+2} + F_{n+1}F_{n+2} = F_n^2 + F_{n+1}F_{n+2}$. A careful study of (a) and (c) imply that each entry of the nine vectors can be written as sums, as well as products, of Fibonacci numbers. This follows from (ii)-(v) in the following lemma.

Lemma 5.1. *Let F_n denote the nth Fibonacci number. Then the following items hold.*

(i) $(F_nF_{n+1})^2 + (F_nF_{n+2})^2 + (F_{n+1}F_{n+2})^2 = (F_n^2 + F_{n+1}F_{n+2})^2$.

(ii) $F_{2n-3}F_{2n-2} = F_1 + F_5 + \cdots + F_{4n-7}$.

(iii) $F_{2n-3}F_{2n-1} = 1 + F_2 + F_6 + \cdots + F_{4n-6}$.

(iv) $F_{2n-2}F_{2n-1} = F_3 + F_7 + \cdots + F_{4n-5}$.
(v) $F_{2n-2}F_{2n} = F_4 + F_8 + \cdots + F_{4n-4}$.

5.1. **Discussion.** Lemma 5.1 (i) can be viewed as a generalization of Raine's results on Pythagorean triples. Statements (i)-(v) are well-known. Since they are not included in [5] or [9], we include them here for completeness. Proofs of (i)-(v) are not included here since they can be easily proved. The identities in Lemma 5.1 come from observations of trivalent graphs. We believe that the recurrence relations (1.1), (5.1) and the trivalent graph $F(e_1, e_2, e_3)$ make these identities *visible*.

To one's surprise, the trivalent graph actually tells us more.

(i) The sum of the first entries (starting from e_4) of the first $2k-1$ consecutive vectors is the negative of a perfect square of a Fibonacci number.
(ii) The sum of the second entry (starting from e_4) of the first k vectors is a product of two Fibonacci numbers.
(iii) The entries of every vector are a product of two Fibonacci numbers. Also, if $(-a, b, c)$ is such a vector, then $c - b - a = \pm 1$.
(iv) Consider two consecutive vectors from the top half of the trivalent graph (for example (e_2, e_4), $(e_4, e_6), \ldots$). Label them as $(-a, b, c)$ and $(-A, B, C)$. Then $C - c = (B - b) + (A - a)$.
(v) Consider two consecutive vectors from the top half (likewise the bottom half) and label them as $(-a, b, c)$ and $(-C, B, A)$. One *sees* that all the entries are the product of two Fibonacci numbers and the product of a and A is one less than a fourth power of a Fibonacci number! For example, $1 \cdot 15 = 2^4 - 1$, $6 \cdot 104 = 5^4 - 1$, $2 \cdot 40 = 3^4 - 1$, $15 \cdot 273 = 12^4 - 1$.

Statements (i)-(v) actually give five well-known identities. For example, (v) shows that *a fourth power of a Fibonacci number minus 1 is the product of four Fibonacci numbers.* Also, the remarkable Gelin-Cesàro Identity

$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1 \tag{5.5}$$

is *visible*. We are currently investigating the trivalent graph $F(u, v, w)$ for arbitrary triples $(u, v.w)$. It turns out that such a study makes many identities *visible*. For example, identities (d4)-(d9) in Section 4 can be seen from some trivalent graphs $F(u, v, w)$. See [6] for more detail.

6. Discussion

In this article, we have demonstrated that a simple study of recurrence relation (1.1) results in a unified proof of many known identities in the literature. This suggests that one may group identities together based on certain recurrence relations (if they exist) and study them as a whole. Note that a given function may satisfy more than one recurrence relations ($(-1)^n F_n$ satisfies (i) below and (3.5)). The next recurrence relations for study, we believe, should be

(i) $x(n+2) = -x(n+1) + x(n)$,
(ii) $x(n+3) = -2x(n+2) + 2x(n+1) + x(n)$.

Identities (in Fibonacci numbers) for such recurrence relations are rare but of great importance. To demonstrate this point, one recalls that the right-hand side of the very elegant identity of Melham's ($F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n$, see [8]) satisfies (i) above. And the following attractive identities of Fairgrieve and Gould [1]

$$F_{n-2}F_{n+1}^2 - F_n^3 = (-1)^n F_{n-1}, \tag{6.1}$$

$$F_{n-3}F_{n+1}^3 - F_n^4 = (-1)^n(F_{n-1}F_{n+3} + 2F_n^2). \tag{6.2}$$

also satisfy (i) and (ii) above, respectively. To conclude our discussion, we give the following example which suggests how a new identity can be obtained by the study of recurrence relation (i). Since the right-hand side of (6.1) satisfies (i) above, $x(n) = F_{n-2}F_{n+1}^2 - F_n^3$ satisfies the same recurrence relation. Namely, $x(n+2) = -x(n+1) + x(n)$. With the help of the famous identity $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$, one finds that

$$F_nF_{n+3}^2 + F_{n-1}F_{n+2}^2 - F_{n-2}F_{n+1}^2 = F_{3n+3}, \tag{6.3}$$

7. Appendix A

Let $u = (u_i)$, $v = (v_i)$, $w = (w_i)$ be 3-tuples with integer entries. We call $\{u, v, w\}$ an $\mathcal{F}$-triple if $N(u) = u_1 + u_2 + u_3$, $N(v) = v_1 + v_2 + v_3$, and $N(w) = w_1 + w_2 + w_3$ are squares and

(i) $2u \cdot v - v \cdot w - w \cdot u = 2N(u)N(v) - N(v)N(w) - N(w)N(u)$,
(ii) $2u \cdot w - v \cdot w - v \cdot u = 2N(u)N(w) - N(v)N(w) - N(v)N(u)$,
(iii) $2v \cdot w - v \cdot u - w \cdot u = 2N(v)N(w) - N(v)N(u) - N(w)N(u)$,

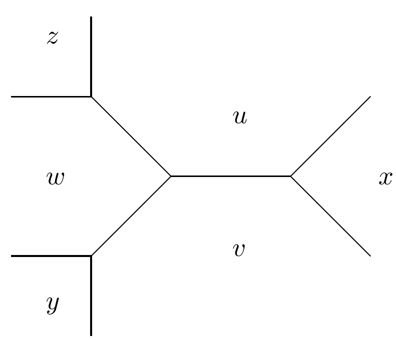
where $u \cdot v$ is the usual dot product. One easily sees that $\{e_1, e_2, e_3\}$ is an $\mathcal{F}$-triple, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The following lemma shows that if $\{u, v, w\}$ is an $\mathcal{F}$-triple, then any vector (a, b, c) in $F(u, v, w)$ has the property $N((a, b, c)) = a + b + c$. This proves (b) of Section 5.

Lemma A. *Let $u = (u_i)$, $v = (v_i)$, $w = (w_i)$ be a $\mathcal{F}$-triples and let $x = (x_i) = 2(u + v) - w$, $y = (y_i) = 2(w + v) - u$, $z = (z_i) = 2(w + u) - v$. Then the following hold.*

(i) $\{u, v, x\}$, $\{u, w, z\}$, and $\{v, w, y\}$ are $\mathcal{F}$-triples,
(ii) $N(x)^2 = (2N(u) + 2N(v) - N(w))^2$, $N(y)^2 = (2N(w) + 2N(v) - N(u))^2$, $N(z)^2 = (2N(w) + 2N(u) - N(v))^2$.

Proof. The proof of the lemma is straightforward by direct calculation. □

Note that x, y, and z in the above lemma are defined as in (5.1) and can be described as follows:



Following our lemma, one may extend the above graph to an infinite trivalent graph that consists of the entire xy-plane such that each triple $\{r, s, t\}$ associated with a vertex is an $\mathcal{F}$-triple. In particular, the entries of every vector of this trivalent graph give a solution to $x^2 + y^2 + z^2 = (x + y + z)^2$. Note that a complete set of integral solutions of the above mentioned equation is given by $\{(mn, m(m + n), n(m + n))\}$.

8. Appendix B

Let $x(n)$ be a function defined on the integers. Consider the equation

$$x(n+k) = a_{k-1}x(n+k-1) + \cdots + a_1 x(n+1) + a_0 x(n). \tag{B1}$$

One easily sees that whether $x(n)$ satisfies some linear recurrence relation depends on whether there exists some k and a_i's such that (B1) holds for all n. In the case $x(n)$ indeed admits some linear recurrence relation, such a relation can be obtained by solving a system of linear equations.

8.1. The recurrence relation $x(n+4) = 3x(n+3) + 6x(n+2) - 3x(n+1) - x(n)$.

In [8], Melham proved that

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n. \tag{B2}$$

We give an alternative proof as follows.

Let $A(n) = F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3$ and $B(n) = (-1)^n F_n$. Note that $A(n)$ and $B(n)$ satisfy the recurrence relation $x(n+4) = 3x(n+3) + 6x(n+2) - 3x(n+1) - x(n)$ and $A(n) = B(n)$ for $n = 0, 1, 2$ and 3. By Lemma 2.2, we may conclude that $A(n) = B(n)$. This completes the proof of (B2). The identity $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$ and Fairgrieve and Gould's identities ((11), (12) of [1]) can be proved by the same method.

8.2. Recurrence relation for F_n^4.

F_n^4 satisfies the recurrence relation

$$x(n+5) = 5x(n+4) + 15x(n+3) - 15x(n+2) - 5x(n+1) + x(n). \tag{B3}$$

One can easily see that both the left- and right-hand side of (6.2) satisfy (B3). Therefore, identity (6.2) can be proved using our technique in subsection 8.1.

8.3. Construction of identities.

Recurrence relations can be used to construct identities. For example, one can actually construct (6.3) as follows.

$x(n)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$
F_{3n+3}	2	8	34	144
$F_nF_{n+3}^2$	0	9	25	128
$F_{n-1}F_{n+2}^2$	1	0	9	25
$F_{n-2}F_{n+1}^2$	-1	1	0	9

Since $F_{3n+3}, F_nF_{n+3}^2, F_{n-1}F_{n+2}^2$ and $F_{n-2}F_{n+1}^2$ satisfy the recurrence relation $x(n+4) = 3x(n+3) + 6x(n+2) - 3x(n+1) - x(n)$, one sees from the above table that

$$F_{3n+3} = F_nF_{n+3}^2 + F_{n-1}F_{n+2}^2 - F_{n-2}F_{n+1}^2. \tag{B4}$$

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