A NOTE ON PERFECT TILINGS OF RECTANGLES WITH RECTANGLES

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Abstract. It is shown that, for every $n \in \{3, 4, \ldots\}$, every rectangle R can be dissected into n rectangles that are mutually similar, but of different size. For the case $n = 2$, a partition of that kind exists if and only if the quotient of the edge lengths of R is larger than 2.

Given a rectangle R of edge lengths a and b and an integer $n \geq 2$, under what conditions on a, b, and n, can the rectangle R be partitioned into n pairwise similar rectangles that are all of different size?[1]

This problem is related to the theory of tilings of polygons $P \subseteq \mathbb{R}^2$. Here a *tiling* of P is a cover of P with polygons $P_i \subseteq P$, $i = 1, \ldots, n$, that have mutually disjoint interiors and are pairwise similar. The tiling is called *perfect* if the pieces P_i are of mutually different size [1]. For a more general insight into the theory of tilings we refer to the comprehensive work [5]. We shall use the notation $R(a, b)$ for a rectangle with edges of lengths a and b. The *eccentricity* c of $R(a, b)$ is the quotient of the longer and the smaller edge length, $c = \max\left\{\frac{a}{b}, \frac{b}{a}\right\} \geq 1$. Two rectangles are similar if and only if their eccentricities agree.

The authors of [4] give algebraic characterizations of all pairs of reals $c_1, c_2 \geq 1$ such that a rectangle of eccentricity c_1 can be tiled with rectangles of eccentricity c_2. But even if R_1 is known to admit tilings with similar copies of R_2, it is not trivial to find all n such that R_1 has a *perfect* tiling with *exactly* n images of R_2. For example, a first perfect tiling of a square with squares is published in 1939 and consists of 55 pieces [8]. Since 1978 it is known that the minimal number of pieces in a perfect tiling of a square with squares is 21 [3].

In the above problem only the eccentricity c of R and the number n of pieces is given, whereas the common eccentricity of the tiles is not restricted. This flexibility is the reason for a surprisingly simple answer to our question.

Theorem. *Let $c \geq 1$ be a real number and $n \geq 2$ be an integer. A rectangle of eccentricity c admits a perfect tiling into n rectangles if and only if either $n = 2$ and $c > 2$ or $n \geq 3$ and c is arbitrary.*

Without any doubt there exists a multitude of different tilings that prove the theorem.[2] Our approach is essentially based on a Fibonacci-type construction. The *Fibonacci polynomials* $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, are defined recursively by

$$f_1(x) = 1, \quad f_2(x) = x \quad \text{and} \quad f_{n+2}(x) = x f_{n+1}(x) + f_n(x),$$

[1] This question has been raised by O. Kalenda during the 41st Winter School in Abstract Analysis held in Kácov, Czech Republic, on January 12–19, 2013. The problem originated in considerations of the design of a story cake on the occasion of the seventh birthday of O. Kalenda's daughter. The number $n = 7$ of stories and the dimensions a and b of the baking try were given. The stories were required to be of different sizes, but to have similar layouts.

[2] A different proof was given by J. Jelínek at the same conference.

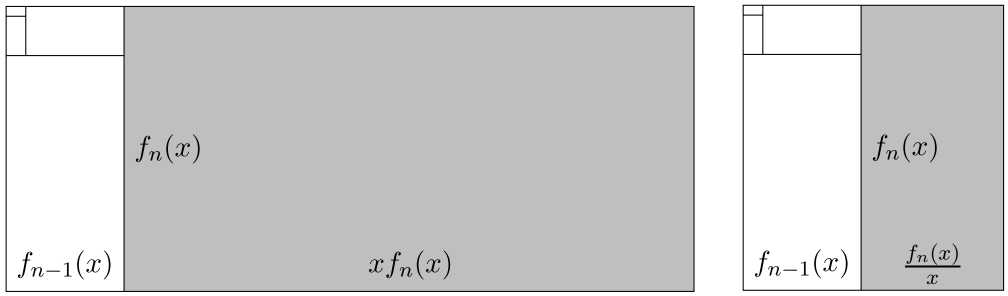

FIGURE 1. Tilings from the proofs of the lemma (left-hand side) and the theorem (right-hand side) with $n = 5$ and $x = 2$. The difference is emphasized.

see [9] and [10, p. 633]. They are closely related with the Fibonacci numbers

$$F_1 = F_2 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

Lemma.

(i) *For every* $n \in \{1, 2, \ldots\}$, f_n *is a polynomial of degree* $n-1$ *with non-negative coefficients and* $f_n(1) = F_n$.

(ii) *For every* $x > 1$, *the sequence* $(f_n(x))_{n=1}^{\infty}$ *is strictly increasing.*

(iii) *For every* $x > 0$ *and every* $n \in \{2, 3, \ldots\}$, *a rectangle* $R(f_n(x), f_{n+1}(x))$ *can be dissected into* n *rectangles* $R(f_i(x), xf_i(x))$, $i = 1, \ldots, n$. *If* $x > 1$, *this tiling is perfect and all edges of all tiles are strictly smaller than* $f_{n+1}(x)$.

Proof. Claims (i) and (ii) follow immediately from the definition of Fibonacci polynomials.

The proof of the first part of (iii) is by induction on n. For $n = 2$, $R(f_2(x), f_3(x)) = R(x, x^2 + 1)$ is split into $R(1, x) = R(f_1(x), xf_1(x))$ and $R(x, x^2) = R(f_2(x), xf_2(x))$. For $n \geq 3$, we cut $R(f_n(x), f_{n+1}(x)) = R(f_n(x), xf_n(x) + f_{n-1}(x))$ first into $R(f_n(x), xf_n(x))$ and $R(f_{n-1}(x), f_n(x))$ and then dissect the piece $R(f_{n-1}(x), f_n(x))$ into $R(f_i(x), xf_i(x))$, $i = 1, \ldots, n-1$, according to the induction hypothesis (see the left-hand part of Figure 1).

Now suppose that $x > 1$. All n pieces $R(f_i(x), xf_i(x))$, $i = 1, \ldots, n$, have eccentricity x and, by (ii), are of different size. Therefore we have a perfect tiling. Moreover, the largest edge of the largest piece has length $xf_n(x)$, which is smaller than $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$. □

The above Fibonacci-type tiling for the case $x = 1$ is already illustrated in [2]; for closely related constructions see [7, 6] and [5, p. 79].

Corollary. *Let* $n \geq 2$ *be an integer. If the eccentricity* c *of a rectangle* R *is strictly larger than* $\frac{F_{n+1}}{F_n}$, *then* R *admits a perfect tiling into* n *rectangles.*

Proof. Let $x > 1$. By claim (iii) of the lemma, the rectangle $R(x) = R(f_n(x), f_{n+1}(x))$ has a perfect tiling into n rectangles. It suffices to show that the eccentricity $c(x) = \frac{f_{n+1}(x)}{f_n(x)}$ of $R(x)$ coincides with c for some $x > 1$. However, this is a consequence of the intermediate value theorem, since

$$c(1) = \frac{F_{n+1}}{F_n} < c < \infty = \lim_{x \to \infty} c(x)$$

by claim (i) of the lemma. □

Note that the above construction does not necessarily give perfect tilings if $x \leq 1$. For example, the lengths $f_1(x)$ and $f_2(x)$ coincide if $x = 1$ and $f_1(x) = f_4(x)$ for $x = 0.45339765\ldots$. We use a slight modification for proving the theorem.

Proof of the Theorem. First let $n = 2$. If a rectangle R can be perfectly tiled with two rectangles R_1 and R_2 of eccentricity $x \geq 1$, then R_1 and R_2 touch each other along a common edge, say of length l, their other edge lengths are $\frac{l}{x}$ and xl, respectively, and perfectness yields $x \neq 1$. The edge lengths of R are l and $\frac{l}{x} + xl$ and its eccentricity is $c = \frac{1}{x} + x$. Since every $x > 1$ is allowed, the set of all attainable eccentricities c consists of all reals strictly larger than 2.

Now we consider $n \geq 3$. Let $x > 1$. We divide a rectangle $R(x) = R\left(f_{n-1}(x) + \frac{f_n(x)}{x}, f_n(x)\right)$ into $R\left(\frac{f_n(x)}{x}, f_n(x)\right)$ and $R(f_{n-1}(x), f_n(x))$. Then we apply claim (iii) of the lemma to the rectangle $R(f_{n-1}(x), f_n(x))$ for dissecting it into $R(f_i(x), xf_i(x))$, $i = 1, \ldots, n-1$ (see the right-hand part of Figure 1). The resulting tiling of $R(x)$ into n rectangles of eccentricity x is perfect, because $R\left(\frac{f_n(x)}{x}, f_n(x)\right)$ has an edge of length $f_n(x)$, which is larger than all edges of the other $n-1$ pieces by part (iii) of the lemma.

The ratio $r(x) = \frac{f_n(x)}{f_{n-1}(x) + \frac{f_n(x)}{x}}$ of the edge lengths of $R(x)$ satisfies

$$r(1) = \frac{F_n}{F_{n-1} + F_n} < 1 \leq c < \infty = \lim_{x \to \infty} r(x)$$

according to claim (i) of the lemma. Hence there exists $x_0 > 1$ such that $R(x_0)$ has the required eccentricity $r(x_0) = c$. This completes the proof. □

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