A NON-LINEAR IDENTITY FOR A PARTICULAR CLASS OF POLYNOMIAL FAMILIES

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Abstract. We state and prove a new non-linear identity for a class of polynomial families associated with integer sequences whose ordinary generating functions have quadratic governing equations with functional (polynomial) coefficients.

1. Introduction

Any sequence of integers whose (ordinary) generating function $T(x)$ satisfies the quadratic equation

$$0 = U(x)T^2(x) + V(x)T(x) + W(x) \tag{1.1}$$

with functional coefficients $U(x), V(x), W(x) \in \mathbb{Z}[x]$, can be considered to give rise naturally to a family of associated polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, whose general form is defined as

$$\begin{aligned} \alpha_n(x) &= \alpha_n(U(x), V(x), W(x)) \\ &= (1,0)\begin{pmatrix} -V(x) & U(x) \\ -W(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0. \end{aligned} \tag{1.2}$$

The first few polynomials are, explicitly,

$$\begin{aligned} \alpha_0(x) &= 1, \\ \alpha_1(x) &= -V(x), \\ \alpha_2(x) &= V^2(x) - U(x)W(x), \\ \alpha_3(x) &= 2U(x)V(x)W(x) - V^3(x), \\ \alpha_4(x) &= V^4(x) - 3U(x)V^2(x)W(x) + U^2(x)W^2(x), \\ \alpha_5(x) &= 4U(x)V^3(x)W(x) - 3U^2(x)V(x)W^2(x) - V^5(x), \end{aligned} \tag{1.3}$$

etc., with general closed form [1, Eq. (52), p. 24]

$$\alpha_n(x) = \frac{1}{2^{n+1}} \frac{[-V(x) + \rho(x)]^{n+1} - [-V(x) - \rho(x)]^{n+1}}{\rho(x)}, \tag{1.4}$$

$\rho(x) = \rho(U(x), V(x), W(x)) = \sqrt{V^2(x) - 4U(x)W(x)}$ being the 'discriminant' function for (1.1); the polynomials have [1, Lemma 3, p. 22] a generating function $(U(x)W(x)y^2 + V(x)y + 1)^{-1} = \sum_{n \geq 0} \alpha_n(x)y^n$. We assume non-zero arguments $U(x), V(x), W(x)$ of $\alpha_n(x)$ throughout the paper.

In previous work identities for three particular instances of polynomial families have been given—so called 'auto-identities' in [1, 4] (developed through the analysis and computerized algebraic implementation of low-order cases of (numeric) Householder root finding algorithms), and in [2] where identities of a different nature (involving polynomial derivatives) can be found:

$$\begin{aligned}\text{Catalan Poly. Family}: \ P_n(x) &= \alpha_n(x,-1,1),\\ \text{(Large) Schröder Poly. Family}: \ S_n(x) &= \alpha_n(x,x-1,1),\\ \text{Motzkin Poly. Family}: \ M_n(x) &= \alpha_n(x^2,x-1,1). \end{aligned} \tag{1.5}$$

The Catalan, (Large) Schröder and Motzkin sequences are well-known to enumerate different types of 2D lattice paths and in this sense form a natural small grouping of sequences (OEIS Sequence Nos. A000108, A006318 and A001006, resp. [6]). There are, of course, many dozens of important integer sequences whose generating functions satisfy an equation of type (1.1) and with whom each has an associated polynomial family $\alpha_n(U(x),V(x),W(x))$.

In this short paper we derive an interesting non-linear recurrence identity satisfied by the general family of polynomials $\alpha_n(x) = \alpha_n(U(x),V(x),W(x))$.

2. Identity and Proof

We give the following identity.

Identity. For $n \geq 1$,

$$U^n(x)W^n(x) = \alpha_n^2(x) + U(x)W(x)\alpha_{n-1}^2(x) + V(x)\alpha_n(x)\alpha_{n-1}(x).$$

Proof. We begin with the following result.

Lemma 2.1. *For $n \geq 1$,*

$$\frac{1}{T^n(x)} = \frac{\alpha_n(x) - U(x)\alpha_{n-1}(x)T(x)}{W^n(x)}.$$

Proof. Consider the quadratic equation (1.1) for $T(x)$. Setting

$$\hat{T}(x) = \frac{W(x)}{U(x)T(x)}, \tag{L.1}$$

it is easy to see that (1.1) is also satisfied by $\hat{T}(x)$. Thus, since

$$T^n(x) = \frac{\alpha_{n-1}(x)T(x) - W(x)\alpha_{n-2}(x)}{U^{n-1}(x)} \tag{L.2}$$

from other work [1, Eq. (55), p. 26], we can write

$$\hat{T}^n(x) = \frac{\alpha_{n-1}(x)\hat{T}(x) - W(x)\alpha_{n-2}(x)}{U^{n-1}(x)}, \tag{L.3}$$

from which, using (L.1), we obtain

$$\frac{1}{T^n(x)} = \frac{1}{W^{n-1}(x)T(x)}f(x), \tag{L.4}$$

where

$$f(x) = \alpha_{n-1}(x) - U(x)\alpha_{n-2}(x)T(x). \tag{L.5}$$

Now denote by $\mathbf{M}(x)$ the matrix

$$\mathbf{M}(x) = \begin{pmatrix} -V(x) & U(x) \\ -W(x) & 0 \end{pmatrix} \tag{L.6}$$

which is integral to the definition (1.2), so that $\alpha_n(x) = (1,0)\mathbf{M}^n(x)\binom{1}{0}$, $n \geq 0$. Then, on observing that the identity

$$\mathbf{0}_2 = U(x)W(x)\mathbf{I}_2 + V(x)\mathbf{M}(x) + \mathbf{M}^2(x) \tag{L.7}$$

holds (where $\mathbf{0}_2$ and $\mathbf{I}_2$ are the respective (2×2) zero and identity matrices), post multiplication throughout (L.7) by $\mathbf{M}^n(x)\binom{1}{0}$ readily yields[1]

$$0 = U(x)W(x)\alpha_n(x) + V(x)\alpha_{n+1}(x) + \alpha_{n+2}(x), \qquad n \geq 0, \tag{L.8}$$

and from (L.5) we can write, using (1.1) as appropriate,

$$\begin{aligned} f(x) &= \alpha_{n-1}(x) - U(x)\left(\frac{-[V(x)\alpha_{n-1}(x) + \alpha_n(x)]}{U(x)W(x)}\right)T(x) \\ &= \frac{1}{W(x)}\{\alpha_n(x)T(x) + [V(x)T(x) + W(x)]\alpha_{n-1}(x)\} \\ &= \frac{T(x)}{W(x)}[\alpha_n(x) - U(x)\alpha_{n-1}(x)T(x)], \end{aligned} \tag{L.9}$$

substitution of which into (L.4) delivers the result. □

The Identity follows quickly, for combining (L.2) with Lemma 2.1 gives

$$\begin{aligned} T(x) &= T^{n+1}(x)\frac{1}{T^n(x)} \\ &= \left(\frac{\alpha_n(x)T(x) - W(x)\alpha_{n-1}(x)}{U^n(x)}\right)\left(\frac{\alpha_n(x) - U(x)\alpha_{n-1}(x)T(x)}{W^n(x)}\right) \\ &= \frac{T(x)[\alpha_n^2(x) + U(x)W(x)\alpha_{n-1}^2(x) + V(x)\alpha_n(x)\alpha_{n-1}(x)]}{U^n(x)W^n(x)}, \end{aligned} \tag{I.1}$$

having again used (1.1); this is the Identity when rearranged. □

The following remarks complete the paper, giving the reader a better context for the work.

Remark 2.1. With $G(x) = (1-\sqrt{1-4x})/(2x)$, $S(x) = (1-x-\sqrt{1-6x+x^2})/(2x)$ and $M(x) = (1-x-\sqrt{1-2x-3x^2})/(2x^2)$ denoting the respective generating functions for the Catalan, (Large) Schröder and Motzkin sequences, Lemma 2.1 specializes to read, for $n \geq 1$,

$$\frac{1}{G^n(x)} = P_n(x) - xP_{n-1}(x)G(x), \tag{2.1}$$

and

$$\begin{aligned} \frac{1}{S^n(x)} &= S_n(x) - xS_{n-1}(x)S(x), \\ \frac{1}{M^n(x)} &= M_n(x) - x^2M_{n-1}(x)M(x). \end{aligned} \tag{2.2}$$

With reference to the group (1.5), we have included these results for consistency with some of our earlier work. We note, for completeness, that expressions for integer powers and inverse powers of the Catalan generating function have been considered by W. Lang. Under

[1] As a point of interest, the recurrence (L.8) was seen previously as [1, Lemma 1.1, p. 10], and established in an alternative fashion.

OEIS Sequence No. A115139 (whose terms are described by successive coefficients of Catalan polynomials) the formula (2.1) for $1/G^n(x)$, and the Catalan specialization $G^n(x) = x^{1-n}[P_{n-1}(x)G(x) - P_{n-2}(x)]$ of (L.2), are both posted by Lang, with further details seen in his 2000 publication [5].

Remark 2.2. The Catalan instance of (L.2) gives immediately that, for all $n \geq 0$, the ratio $P_n(x)/P_{n+1}(x)$ of Catalan polynomials is an order $(\lfloor \frac{1}{2}n \rfloor, \lfloor \frac{1}{2}(n+1) \rfloor)$ Padé approximate of the Catalan sequence generating function $G(x)$ (see the result and an interesting alternative proof on pp. 89–90 of [4], and also Remark 4 on p. 92); we have been able to make corresponding statements in relation to (Large) Schröder and Motzkin polynomials, and their respective generating functions, in [1, Section 5.1, pp. 27–28].

Remark 2.3. We mention that in [3] the $W(x) = 1$ version of the basic governing equation (1.1) was treated and, where $\alpha_n(x) = \alpha_n(U(x), V(x), 1)$, the simplified recurrence $U^n(x) = \alpha_n^2(x) + U(x)\alpha_{n-1}^2(x) + V(x)\alpha_n(x)\alpha_{n-1}(x)$ established.[2] The Catalan, (Large) Schröder and Motzkin cases of our Identity here—for which each has $W(x) = 1$—are to be found therein as equations (6), (P1), and (7) (see pp. 41, 44), with the Schröder recurrence used as the basis of one for the Delannoy numbers intimately connected to them. We note furthermore that it is in fact possible to obtain our fully general Identity directly from this reduced instance for which $W(x) = 1$ (see the Appendix for completeness). This, however, does not diminish our completely independent formulation here.

Remark 2.4. As a final remark, we note that a further new identity arises with minimal effort—the recursion (L.8), when written $0 = U(x)W(x)\alpha_{n-1}(x) + V(x)\alpha_n(x) + \alpha_{n+1}(x)$, can be combined with our Identity to give $U^n(x)W^n(x) = \alpha_n^2(x) - \alpha_{n-1}(x)\alpha_{n+1}(x)$ $(n \geq 1)$ trivially, with $V(x)$ absent.

3. Summary

In this short paper we have formulated a new non-linear identity for families of polynomials associated with integer sequences whose generating functions have quadratic governing equations, adding to recent work on the families thus described. Such sequences are many and varied, and the polynomials they produce have interesting mathematical properties (which have been reported elsewhere); the recurrence Identity established here serves as a case in point and, as seen from Remark 2.4, the potential to develop further results for $\alpha_n(U(x), V(x), W(x))$ is evident.

On a cautionary note, analysis which accommodates polynomial families associated with a general cubic generating function governing equation for sequences is conceptually problematic insofar as it is not clear how to actually define, in any meaningful way, a 3×3 matrix (equivalent to $\mathbf{M}(x)$ here) which contains the four characterizing functional coefficients present in such a cubic equation.

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[2] See the Theorem on p. 41. In that paper, of course—being characterized only by $U(x), V(x)$— the general polynomial $\alpha_n(x)$ was denoted $\alpha_n(U(x), V(x))$. This recursion in [3] for $\alpha_n(U(x), V(x))$ was formulated in two (matrix based) ways, each different from the derivation of our Identity for $\alpha_n(U(x), V(x), W(x))$.

Appendix

Consider the quadratic in $T(x)$ (1.1) characterizing through (1.2) the family of polynomials $\alpha_n(x) = \alpha_n(U(x), V(x), W(x))$. Dividing throughout by $W(x)$ (assumed non-zero) it reads

$$0 = U^*(x)T^2(x) + V^*(x)T(x) + 1, \tag{A.1}$$

where

$$U^*(x) = \frac{U(x)}{W(x)}, \qquad V^*(x) = \frac{V(x)}{W(x)}. \tag{A.2}$$

Now we know, as stated in Remark 2.3, that the polynomial family $\alpha_n^*(x) = \alpha_n^*(U^*(x), V^*(x))$ satisfies from [3] the recurrence

$$U^{*n}(x) = \alpha_n^{*2}(x) + U^*(x)\alpha_{n-1}^{*2}(x) + V^*(x)\alpha_n^*(x)\alpha_{n-1}^*(x), \tag{A.3}$$

so that, using (A.2) and additionally the relation

$$\alpha_n^*(x) = \frac{\alpha_n(x)}{W^n(x)} \tag{A.4}$$

immediate from (1.2), then (A.3) becomes—in terms of $U(x), V(x), W(x)$ and $\alpha_n(x)$—precisely our Identity.

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