ODD FIBBINARY NUMBERS AND THE GOLDEN RATIO

LINUS LINDROOS, ANDREW SILLS, AND HUA WANG

ABSTRACT. The fibbinary numbers are positive integers whose binary representation contains no consecutive ones. We prove the following result: If the jth odd fibbinary is the nth *odd* fibbinary number, then $j = \lfloor n\phi^2 \rfloor - 1$.

1. BACKGROUND

The Fibonacci numbers $\{F_n\}_{n \geq 0}$ are given by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$.

Recall the following theorem of Zeckendorf [1].

Zeckendorf's Theorem. *Every positive integer can be written uniquely as the sum of distinct, nonconsecutive Fibonacci numbers.*

The *Zeckendorf representation* $z(n)$ of $n \in \mathbb{N}$ is the unique k-tuple of decreasing nonconsecutive Fibonacci numbers whose sum is n. Note that although $F_2 = F_1 = 1$, we always associate 1 with F_2 in the Zeckendorf representation.

For example,

$$z(4) = (3, 1) = (F_4, F_2),$$
$$z(5) = (5) = (F_5),$$
$$\text{and} \quad z(100) = (89, 8, 3) = (F_{11}, F_6, F_4).$$

The sequence $\{\text{fib}(n)\}_{n \geq 1}$ of *fibbinary numbers*[1] is given as follows: For $n > 0$, if $z(n) = (F_{i_1}, F_{i_2}, \dots F_{i_k})$ is the Zeckendorf representation of n, then

$$\text{fib}(n) := \sum_{j=1}^{k} 2^{i_j - 2}.$$

For example,

$$\text{fib}(4) = \text{fib}(F_4 + F_2) = 2^{4-2} + 2^{2-2} = 101_2 = 5.$$
$$\text{fib}(5) = \text{fib}(F_5) = 2^{5-3} = 1000_2 = 8.$$
$$\text{fib}(100) = \text{fib}(F_{11} + F_6 + F_4) = 2^{11-2} + 2^{6-2} + 2^{4-2} = 1000010100_2 = 532,$$

where the 2 subscript indicates the usual binary (base two) representation.

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[1]According to [3], the name "fibbinary" is due to Marc LeBrun: "... integers whose binary representation contains no consecutive ones and noticed that the number of such numbers with n bits was F_n."—posting to `sci.math` by Bob Jenkins, July 17, 2002.

2. Statement of Main Result Relating Odd Fibbinaries to the Golden Ratio

It is easy to generate the odd fibbinary numbers from the binary representations and find the corresponding value of the original integer, e.g.,

- $1_2 = 2^0 = \text{fib}(F_2) = \text{fib}(1)$;
- $101_2 = 2^2 + 2^0 = \text{fib}(F_4 + F_2) = \text{fib}(4)$;
- $1001_2 = 2^3 + 2^0 = \text{fib}(F_5 + F_2) = \text{fib}(6)$;
- ....

Let $\text{odfib}(n)$ denote the nth odd fibbinary number, i.e.,

$$\text{odfib}(1) = 1_2, \text{odfib}(2) = 101_2, \text{odfib}(3) = 1001_2, \ldots.$$

Let

$$Z(n) := \text{fib}^{-1}(\text{odfib}(n)),$$

so that

$$Z(1) = 1, Z(2) = 4, Z(3) = 6, Z(4) = 9, \ldots.$$

In other words, if the nth odd fibbinary number is the jth fibbinary number, then $j = Z(n)$.

This sequence

$$1, 4, 6, 9, 12, 14, 17, \ldots$$

appears to be "A003622" in OEIS [2], defined as

$$\{\lfloor n\phi^2 \rfloor - 1\}_{n=1}^\infty = \{\lfloor n\phi \rfloor + n - 1\}_{n=1}^\infty \tag{1}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio, which satisfies $\phi^2 = \phi + 1$, and of course arises as

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.$$

j	$z(j)$	$\text{fib}(j)$	$\text{odfib}^{-1}(j)$
$1 = \lfloor \phi^2 \rfloor - 1 = Z(1)$	$(1) = (F_2)$	$1_2 = 1$	1
2	$(2) = (F_3)$	$10_2 = 2$	—
3	$(3) = (F_4)$	$100_2 = 4$	—
$4 = \lfloor 2\phi^2 \rfloor - 1 = Z(2)$	$(3, 1) = (F_4, F_2)$	$101_2 = 5$	2
5	$(5) = (F_5)$	$1000_2 = 8$	—
$6 = \lfloor 3\phi^2 \rfloor - 1 = Z(3)$	$(5, 1) = (F_5, F_2)$	$1001_2 = 9$	3
7	$(5, 2) = (F_5, F_3)$	$1010_2 = 10$	—
8	$(8) = (F_6)$	$10000_2 = 16$	—
$9 = \lfloor 4\phi^2 \rfloor - 1 = Z(4)$	$(8, 1) = (F_6, F_2)$	$10001_2 = 17$	4
10	$(8, 2) = (F_6, F_3)$	$10010_2 = 18$	—
11	$(8, 3) = (F_6, F_4)$	$10100_2 = 20$	—
$12 = \lfloor 5\phi^2 \rfloor - 1 = Z(5)$	$(8, 3, 1) = (F_6, F_4, F_2)$	$10101_2 = 21$	5

The correspondence displayed above is naturally conjectured to be true in general.

Theorem 2.1. *Let j be a positive integer such that the jth fibbinary number is odd. Suppose that this jth fibbinary number is the nth odd fibbinary number. Then*

$$j = Z(n) = \lfloor n\phi^2 \rfloor - 1 = \lfloor n\phi \rfloor + n - 1 \tag{2}$$

for any $n \geq 1$.

3. Proof of Theorem 2.1

It is easy to check that (2) is true for small values of n. In the rest of this note we provide a general proof for $n \geq 3$.

Next, we record the following observation on odfib(n) as a lemma.

Lemma 3.1. *For any $1 \leq k \leq F_{n-1}$ and $n \geq 2$, we have*

$$\text{odfib}(F_n + k) = 2^n + \text{odfib}(k).$$

Proof. The results follow since each digit in the binary representation of odfib(n) corresponds to a specific Fibonacci number. $\square$

For example,

$$\text{odfib}(10) = \text{odfib}(8 + 2) = \text{odfib}(F_6 + 2) = 1000101_2 = 2^6 + 101_2 = 2^6 + \text{odfib}(2).$$

Lemma 3.2. *For any $1 \leq k \leq F_{n-1}$ and $n \geq 2$, we have*

$$Z(F_n + k) = F_{n+2} + Z(k).$$

Proof. Since each digit in the binary representation of odfib(n) corresponds to a distinct Fibonacci number in the sum of $Z(n)$ under the Zeckendorf representation, we can claim the same for $Z(n)$. That is,

$$\begin{aligned}
Z(F_n + k) &= \text{fib}^{-1}(\text{odfib}(F_n + k)) \\
&= \text{fib}^{-1}(2^n + \text{odfib}(k)) \\
&= F_{n+2} + \text{fib}^{-1}(\text{odfib}(k)) \\
&= F_{n+2} + Z(k).
\end{aligned}$$

$\square$

For example,

$$Z(10) = Z(F_6 + 2) = 25 = 21 + 4 = F_8 + Z(2).$$

From Lemma 3.2, we have

$$Z(F_n + k) - Z(F_n + k - 1) = Z(k) - Z(k - 1) \tag{3}$$

for $2 \leq k \leq F_{n-1}$ and

$$Z(F_n + 1) = F_{n+2} + 1. \tag{4}$$

Now to show (2) for any n, we only need to show analogues of (3) and (4) for $\lfloor n\phi^2 \rfloor - 1$, i.e.,

$$\begin{aligned}
\lfloor (F_n + k)\phi \rfloor - \lfloor (F_n + k - 1)\phi \rfloor &= \lfloor (F_n + k)\phi^2 \rfloor - \lfloor (F_n + k - 1)\phi^2 \rfloor \\
&= \lfloor k\phi^2 \rfloor - \lfloor (k - 1)\phi^2 \rfloor \\
&= \lfloor k\phi \rfloor - \lfloor (k - 1)\phi \rfloor \tag{5}
\end{aligned}$$

for $2 \leq k \leq F_{n-1}$, and

$$\lfloor (F_n + 1)\phi \rfloor + F_n = \lfloor (F_n + 1)\phi^2 \rfloor - 1 = Z(F_n + 1). \tag{6}$$

Remark. Intuitively, this can be considered as using $Z(F_n + 1)$ as the "stepping stone" to prove (2) for $Z(F_n + k)$ for $k = 2, 3, \ldots, F_{n-1}$ using induction.

In order to establish (5), it is essentially sufficient to show that $\{F_n\phi\}$ is never large enough to affect the difference $\lfloor k\phi \rfloor - \lfloor (k-1)\phi \rfloor$, where $\{x\} := x - \lfloor x \rfloor$, the fractional part of the real number x.

We make use of the following fact, which is easily established by induction in n:

$$(-1)^n \tau^n = -F_n\tau + F_{n-1}, \tag{7}$$

where $\tau = \frac{\sqrt{5}-1}{2} = \phi - 1$ satisfying $\tau^2 = -\tau + 1$. Consequently,

$$\{F_n\phi\} = \{F_n\tau\} = \{-(-\tau)^n\}$$

for any n.

Making use of the fact that $\tau = \phi - 1$, it suffices to show

$$\lfloor (F_n + k)\tau \rfloor - \lfloor (F_n + k - 1)\tau \rfloor = \lfloor k\tau \rfloor - \lfloor (k-1)\tau \rfloor. \tag{8}$$

To show (8), simply consider

$$\{k\tau\} \pm \{F_n\tau\}$$

for any $1 \le k \le F_{n-1}$. We will show that this value never reaches 1 or goes below zero and hence, $\{F_n\tau\}$ will not affect $\lfloor k\tau \rfloor - \lfloor (k-1)\tau \rfloor$.

(i) To show $\{k\tau\} + \{F_n\tau\} < 1$, consider the Zeckendorf representation of k as the sum of nonconsecutive Fibonacci numbers.

If $k < F_{n-1}$, then

$$k = F_{a_1} + F_{a_2} + \cdots + F_{a_s}$$

where

$$1 \le a_1 \le a_2 - 2 \le a_2 \le a_3 - 2 \le \cdots \le a_s \le n - 2.$$

Then

$$\begin{aligned}
\{k\tau\} + \{F_n\tau\} &= \{(F_{a_1} + \cdots + F_{a_s})\tau\} + \{F_n\tau\} \\
&\le \tau^{a_1} + \tau^{a_2} + \cdots + \tau^{a_s} + \tau^n \\
&< \tau + \tau^3 + \tau^5 + \cdots \\
&= \frac{\tau}{1 - \tau^2} \\
&= 1.
\end{aligned}$$

If $k = F_{n-1}$, we have

$$\{k\tau\} + \{F_n\tau\} = \tau^{n-1} + \tau^n \le \tau^2 + \tau^3 < 1$$

for any $n \ge 3$.

(ii) To show $\{k\tau\} - \{F_n\tau\} > 0$, simply note that

$$\begin{aligned}
\{k\tau\} - \{F_n\tau\} &= \{(F_{a_1} + \cdots + F_{a_s})\tau\} - \{F_n\tau\} \\
&\ge \tau^{a_1} - \tau^{a_2} - \cdots - \tau^{a_s} - \tau^n \\
&> \tau^{a_1} - \tau^{a_1+2} - \tau^{a_1+4} - \cdots \\
&= \tau^{a_1}\left(1 - \frac{\tau^2}{1 - \tau^2}\right) \\
&> 0
\end{aligned}$$

if $k < F_{n-1}$ and

$$\{k\tau\} - \{F_n\tau\} = \tau^{n-1} - \tau^n > 0$$

if $k = F_{n-1}$.

Cases (i) and (ii) imply that

$$\lfloor F_n\tau + k\tau \rfloor = \lfloor F_n\tau \rfloor + \lfloor k\tau \rfloor .$$

Thus (8) and (5) are proved.

By (4), (6) is equivalent to

$$\lfloor (F_n + 1)\phi \rfloor = F_{n+1} + 1. \tag{9}$$

Using (7) it follows that

$$\begin{aligned}
\lfloor (F_n + 1)\phi \rfloor &= F_n + 1 + \lfloor (F_n + 1)\tau \rfloor \\
&= F_n + 1 + \lfloor F_{n-1} - (-\tau)^n + \tau \rfloor \\
&= F_n + 1 + F_{n-1} + \lfloor \tau - (-\tau)^n \rfloor \\
&= F_{n+1} + 1
\end{aligned}$$

for $n \geq 3$. Therefore, (9) is true and hence, (6) is proved. $\qquad\square$

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References

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MSC2010: 11B39

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460
E-mail address: `ll01383@georgiasouthern.edu`

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460
E-mail address: `asills@georgiasouthern.edu`

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460
E-mail address: `hwang@georgiasouthern.edu`