SOME GENERALIZED FIBONACCI IDENTITIES INCLUDING POWERS AND BINOMIAL COEFFICIENTS

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Abstract. We investigate several generalized Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions. We also discuss some identities yielding from Tribonacci, Tetranacci numbers, and even from general Fibonacci s-step numbers.

1. Introduction

Consider the sequence $\{u_n\}$, satisfying the three-term recurrence relation:

$$u_n = au_{n-1} + bu_{n-2} \quad (n \ge 2), \quad u_0 = 0, \quad u_1 = 1\,, \tag{1.1}$$

where a and b are any nonzero integers with $a^2 + 4b > 0$. If $a = b = 1$, then $u_n = F_n$ is the nth Fibonacci number. In [1] and [2] linear recurrence sequences linked to the elements of Pascal triangle, continued fractions expansions of the ratios, and bivariate polynomials are investigated about the sequence $\{u_n\}$. For example, the ratio of the consecutive terms of u_n has the following non-regular continued fraction expansion:

$$\frac{u_n}{u_{n-1}} = a + \cfrac{b}{a + \ddots + \cfrac{b}{a + \cfrac{b}{u_2/u_1}}} = a + \underbrace{\frac{b}{a} + \cdots + \frac{b}{a}}_{n-2}\,.$$

It is known and is easy to prove by induction that

$$u_n = \frac{\Psi^n - \bar{\Psi}^n}{\sqrt{a^2+4b}} \quad (n \ge 0)\,, \tag{1.2}$$

where

$$\Psi = \frac{a + \sqrt{a^2+4b}}{2} = a + \frac{b}{a+}\,\frac{b}{a+}\cdots$$

and

$$\bar{\Psi} = \frac{a - \sqrt{a^2+4b}}{2} = -\frac{b}{\Psi}\,,$$

satisfying $\Psi + \bar{\Psi} = a$, $\Psi\bar{\Psi} = -b$, $\Psi - \bar{\Psi} = \sqrt{a^2+4b}$ and $\Psi^n = \Psi u_n + bu_{n-1}$ $(n \ge 1)$.

In addition, in [1] and [2], by using the method of ordinary power series generating functions, some new identities for u_n are obtained, including

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\Psi}{b}\right)^k u_k = \frac{\Psi}{b}\left(\frac{a\Psi}{b} + 2\right)^{n-1} \qquad (n \ge 1)$$

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and

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\bar{\Psi}}{b}\right)^k u_k = \frac{\bar{\Psi}}{b} \left(\frac{a\bar{\Psi}}{b} + 2\right)^{n-1} \qquad (n \geq 1)\,.$$

Such identities are generalizations of those for Fibonacci numbers F_n in [5].

In this paper, we investigate several generalized Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions. We also discuss some identities yielding from Tribonacci, Tetranacci numbers, and even from general Fibonacci s-step numbers.

2. Ordinary Power Series Generating Functions

In this section, we show some new generalized identities by using the method of the ordinary power series generating functions. Let $\{a_k\}$ and $\{b_k\}$ be sequences with the property that a_k is the finite difference of b_k, that is, $a_k = \Delta b_k := b_{k+1} - b_k$, for $k \geq 0$. Let $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$, and $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$. In [5] Spivey derives expressions for g_n in terms of h_n and for h_n in terms of g_n. According to Spivey [5], the *ordinary power series generating function* $U(z)$ *of* g_n is the generating function of the infinite sum $\sum_{n=0}^{\infty} g_n z^n$. By applying his method, we [1, 2] obtained the following result.

Lemma 2.1. *Let $\{u_k\}$ be a generalize Fibonacci-type sequence defined by $u_n = au_{n-1} + bu_{n-2}$ $(n \geq 2)$ with $u_0 = 0$ and $u_1 = 1$, and c be any real number. Then, the ordinary power series generating function $U(z)$ of $\sum_{k=0} \binom{n}{k} c^k u_k$ is given by*

$$U(z) = \frac{cz}{1 - (ac+2)z - (bc^2 - ac - 1)z^2}\,.$$

Namely,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c^k u_k z^n = \frac{cz}{1 - (ac+2)z - (bc^2 - ac - 1)z^2}\,.$$

Proof. Let $b_k = c^{k+1} u_{k+1}$ and $a_k = c^k u_k$. Let $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$ and $g_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. We have $\delta b_k = b_{k+1} - b_k = c^{k+2} u_{k+2} - c^{k+1} u_{k+1} = (ac-1) b_k + bc^2 a_k$. Solving the system of recurrences $h_{n+1} - 2h_n - c = (ac-1)h_n + bc^2 g_n$ and $g_{n+1} - g_n = h_n$ for g_n, we get $g_{n+2} = (ac+2) g_{n+1} + (bc^2 - ac - 1) g_n + c$. Hence, the ordinary power series generating function $U(z)$ of g_n satisfies the equation $U(z) - (ac+2)zU(z) - (bc^2 - ac - 1)z^2 U(z) = cz$. Solving for $U(z)$ completes the proof. □

By applying Lemma 2.1, we obtain the following theorems below.

Theorem 2.2. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^k u_k = r_n,$$

where r_n $(n \geq 0)$ satisfies the recurrence relation

$$r_n = (ac+2) r_{n-1} + (bc^2 - ac - 1) r_{n-2} \quad (n \geq 2) \tag{2.1}$$

with $r_0 = 0$ and $r_1 = c$.

Remark. Let ξ be the larger root of the quadratic equation $x^2-(ac+2)x-(bc^2-ac-1)=0$, and $\bar{\xi}$ be its conjugate. Namely,

$$\xi=\frac{ac+2+c\sqrt{a^2+4b}}{2} \quad \text{and} \quad \bar{\xi}=\frac{ac+2-c\sqrt{a^2+4b}}{2}.$$

Then r_n $(n\geq 0)$ can be expressed as

$$r_n=\frac{1}{\sqrt{a^2+4b}}(\xi^n-\bar{\xi}^n).$$

If $c=2$ and $a=b=1$, then $u_k=F_k$, and by $F_{3n}=4F_{3n-3}+F_{3n-6}$ $(n\geq 2)$ with $F_0=0=r_0$ and $F_3=2=r_1$ we have $r_n=F_{3n}$. Hence, we can get the known identity

$$\sum_{k=0}^{n}\binom{n}{k}2^kF_k=F_{3n}.$$

Proof. We shall use the function

$$U(z)=\frac{cz}{1-(ac+2)z-(bc^2-ac-1)z^2}$$

in Lemma 2.1 as it is. Let $U(z)$ be expressed as the infinite sum:

$$U(z)=r_0+r_1z+r_2z^2+r_3z^3+\cdots \quad (|z|<1).$$

Since

$$\begin{aligned}&U(z)-(ac+2)zU(z)-(bc^2-ac-1)z^2U(z)\\&=r_0+\big(r_1-(ac+2)r_0-c\big)z+\sum_{n=2}^{\infty}\big(r_n-(ac+2)r_{n-1}-(bc^2-ac-1)r_{n-2}\big)z^n\\&=0,\end{aligned}$$

we have $r_0=0$, $r_1=c$ and $r_n=(ac+2)r_{n-1}+(bc^2-ac-1)r_{n-2}$ $(n\geq 2)$. $\square$

Theorem 2.3. *For $n\geq 0$ we have*

$$\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^ku_k=\lambda_n,$$

where the numbers λ_n $(n\geq 0)$ satisfy the recurrence relation

$$\lambda_n=(ad+2c)\lambda_{n-1}+(bd^2-acd-c^2)\lambda_{n-2} \quad (n\geq 2) \tag{2.2}$$

with $\lambda_0=0$ and $\lambda_1=d$.

Proof. Let c and z be replaced by d/c and cz, respectively in Lemma 2.1. We call this function $V(z)$ in place of $U(z)$. Namely,

$$V(z)=\frac{dz}{1-(ad+2c)-(bd^2-acd-c^2)z^2}.$$

Assume that $V(z)=\sum_{n=0}\lambda_nz^n$ $(|z|<1)$. Since

$$\begin{aligned}&V(z)-(ad+2c)zV(z)=(bd^2-acd-c^2)z^2V(z)-dz\\&=\lambda_0+\big(\lambda_1-(ad+2c)\lambda_0-d\big)z+\sum_{n=2}^{\infty}\big(\lambda_n-(ad+2c)\lambda_{n-1}-(bd^2-acd-c^2)\lambda_{n-2}\big)z^n\\&=0,\end{aligned}$$

we have $\lambda_n - (ad+2c)\lambda_{n-1} - (bd^2 - acd - c^2)\lambda_{n-2} = 0$ $(n \ge 2)$ with $\lambda_0 = 0$ and $\lambda_1 - (ad + 2c)\lambda_0 - d = 0$. Hence, we obtain the desired recurrence relation. $\square$

Let the sequence $\{u_n\}$ be as in Section 1. Let the sequence $\{v_n\}$ satisfy the same three-term recurrence relation as (1.1) with different initial values:

$$v_n = av_{n-1} + bv_{n-2} \quad (n \ge 2), \quad v_0 = 2, \quad v_1 = a. \tag{2.3}$$

If $a = b = 1$, then $v_n = L_n$ is the nth Lucas number. Similarly to the identities about u_n, we have $v_n = \Psi^n + \bar{\Psi}^n$ $(n \ge 0)$ and $\sqrt{a^2+4b}\Psi^n = \Psi v_n + bv_{n-1}$ $(n \ge 1)$.

Corollary 2.4. *For $n \ge 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (2b)^{n-k} a^k u_k = \begin{cases} (a^2+4b)^{\frac{n}{2}} u_n & \text{if } n \text{ is even;} \\ (a^2+4b)^{\frac{n-1}{2}} v_n & \text{if } n \text{ is odd.} \end{cases}$$

Remark. If $a = b = 1$ in Corollary 2.4, then we get

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_k = \begin{cases} 5^{\frac{n}{2}} F_n & \text{if } n \text{ is even;} \\ 5^{\frac{n-1}{2}} L_n & \text{if } n \text{ is odd,} \end{cases}$$

which also appeared in [3, Result 3.1].

There are relations between u_n and v_n. The proof is done by induction and omitted.

Lemma 2.5. *For any integer n we have*

$$u_{n+1} + bu_{n-1} = v_n,$$
$$v_{n+1} + bv_{n-1} = (a^2+4b)u_n.$$

Proof of Corollary 2.4. Let $c = 2b$ and $d = 1$ in Theorem 2.3. Then the number λ_n satisfies the relation $\lambda_n = (a^2+4b)(\lambda_{n-1} - b\lambda_{n-2})$ $(n \ge 2)$. For $n = 0, 1, 2, 3$ we have $\lambda_0 = 0 = u_0$, $\lambda_1 = a = v_1$, $\lambda_2 = a(a^2+4b) = (a^2+4b)u_2$ and $\lambda_3 = a(a^2+4b)(a^2+3b) = (a^2+4b)v_3$. For some n, we assume that

$$\lambda_{2n} = (a^2+4b)^n u_{2n} \quad \text{and} \quad \lambda_{2n+1} = (a^2+4b)^n v_{2n+1}\,.$$

Then by Lemma 2.5 we have

$$\begin{aligned} \lambda_{2n+2} &= (a^2+4b)(\lambda_{2n+1} - b\lambda_{2n}) \\ &= (a^2+4b)\big((a^2+4b)^n v_{2n+1} - b(a^2+4b)^n u_{2n}\big) \\ &= (a^2+4b)^{n+1}(v_{2n+1} - bu_{2n}) \\ &= (a^2+4b)^{n+1} u_{2n+2} \end{aligned}$$

and

$$\begin{aligned} \lambda_{2n+3} &= (a^2+4b)(\lambda_{2n+2} - b\lambda_{2n+1}) \\ &= (a^2+4b)\big((a^2+4b)^{n+1} u_{2n+2} - b(a^2+4b)^n v_{2n+1}\big) \\ &= (a^2+4b)^{n+1}\big((a^2+4b)u_{2n+2} - bv_{2n+1}\big) \\ &= (a^2+4b)^{n+1} v_{2n+3}. \end{aligned}$$

By induction, the proof is done. $\square$

Example 2.6. *Let $c = -1$, $d = 4$, $a = 3$ and $b = -1$ in Theorem 2.3. Then the numbers λ_n satisfy the relation $\lambda_n = 3\lambda_{n-1} - \lambda_{n-2}$ $(n \geq 2)$. Since $u_0 = 0 = F_0$, $u_1 = 1 = F_2$ and $F_{2k} = 3F_{2k-2} - F_{2k-4}$ $(k \geq 2)$, we get $u_k = F_{2k}$. For $n = 0, 1, 2, 3$ we have $\lambda_0 = 0 = F_0$, $\lambda_1 = 4 = L_3$, $\lambda_2 = 40 = 5F_6$ and $\lambda_3 = 420 = 5L = 9$. For some n, we assume that*

$$\lambda_{2n} = 5^n F_{6n} \quad \text{and} \quad \lambda_{2n+1} = 5^n L_{6n+3}.$$

Then by $F_{m+3}+F_{m-3} = L_m F_3 = 2L_m$ ([4, p. 97. no. 56]) and $L_{m+3}+L_{m-3} = 5F_m F_3 = 10F_m$ ([4, p. 91, no. 85])

$$\begin{aligned}
\lambda_{2n+2} &= 10\lambda_{2n+1} - 5\lambda_{2n} \\
&= 10 \cdot 5^n L_{6n+3} - 5 \cdot 5^n F_{6n}) \\
&= 5^{n+1}(2L_{6n+3} - F_{6n}) \\
&= 5^{n+1} F_{6n+6}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{2n+3} &= 10\lambda_{2n+2} - 5\lambda_{2n+1} \\
&= 10 \cdot 5^{n+1} F_{6n+6} - 5 \cdot 5^n L_{6n+3}) \\
&= 5^{n+1}(10F_{6n+6} - L_{6n+3}) \\
&= 5^{n+1} L_{6n+9}.
\end{aligned}$$

By induction, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^{2k} F_{2k} = \begin{cases} 5^{\frac{n}{2}} F_{3n} & \text{if } n \text{ is even;} \\ 5^{\frac{n-1}{2}} L_{3n} & \text{if } n \text{ is odd,} \end{cases}$$

which also appears in [3].

Example 2.7. *Let $c = -1$, $d = 1$, $a = 3$ and $b = -1$ in Theorem 2.3. Then the numbers λ_n satisfy the relation $\lambda_n = \lambda_{n-1} + \lambda_{n-2}$ $(n \geq 2)$ with $\lambda_0 = 0$ and $\lambda_1 = 1$. Thus, $\lambda_n = F_n$. Together with $u_k = F_{2k}$ we get the well-known identity*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{2k} = F_n$$

([4, p. 237, no. 8]).

3. Tribonacci Numbers

Let $F_n^{(3)}$ be Tribonacci numbers defined by $F_n^{(3)} = F_{n-1}^{(3)} + F_{n-2}^{(3)} + F_{n-3}^{(3)}$ $(n \geq 3)$ with $F_0^{(3)} = 0$ and $F_1^{(3)} = F_2^{(3)} = 1$.

Theorem 3.1. *The ordinary power series generating function $T(z)$ of $\sum_{k=0}^{n} \binom{n}{k} c^k F_k^{(3)}$ is given by*

$$T(z) = \frac{cz(1-z)}{1 - (c+3)z - (c^2 - 2c - 3)z^2 - (c^3 - c^2 + c + 1)z^3}.$$

Proof. Let $b_k = c^{k+1}F_{k+1}^{(3)}$ and $h_n = \sum_{k=0}^{n} \binom{n}{k} b_k$, $g_n = \sum_{k=0}^{n} \binom{n}{k} b_{k-1}$ and $f_n = \sum_{k=0}^{n} \binom{n}{k} b_{k-2}$. We have $\Delta b_k = c^{k+2}F_{k+2}^{(3)} - c^{k+1}F_{k+1}^{(3)} = (c-1)b_k + c^2 b_{k-1} + c^3 b_{k-2}$. Solving the system of recurrences $h_{n+1} - 2h_n = (c-1)h_n + c^2 g_n + c^3 f_n$ and $g_{n+1} - g_n = h_n$ for g_n, we obtain $g_{n+2} = (c+2)g_{n+1} + (c^2-c-1)g_n + c^3 f_n$. By $f_{n+1} - f_n = g_n$, we have $f_{n+3} = (c+3)f_{n+2} + (c^2 - 2c-3)f_{n+1} + (c^3 - c^2 + c+1)f_n$. Thus, the ordinary power series generating function $F(z)$ of f_n satisfies the equation $F(z) - (c+3)zF(z) - (c^2-2c-3)z^2F(z) - (c^3-c^2+c+1)z^3F(z) = cz^2$. Solving for $F(z)$, the ordinary power series generating function $F(z)$ of $\sum_{k=1}^{n} \binom{n}{k} c^{k-1}F_{k-1}^{(3)}$ is given by

$$F(z) = \frac{cz^2}{1-(c+3)z-(c^2-2c-3)z^2-(c^3-c^2+c+1)z^3}.$$

Hence, the ordinary power series generating function $T(z) = F(z)(1-z)/z$ of $\sum_{k=0}^{n} \binom{n}{k} c^k F_k^{(3)}$ is given by

$$T(z) = \frac{cz(1-z)}{1-(c+3)z-(c^2-2c-3)z^2-(c^3-c^2+c+1)z^3}.$$

□

By applying Theorem 3.1, we get the following two theorems.

Theorem 3.2. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^k F_k^{(3)} = t_n,$$

where the numbers t_n satisfy the recurrence relation

$$t_n = (c+3)t_{n-1} + (c^2-2c-3)t_{n-2} + (c^3-c^2+c+1)t_{n-3} \quad (n \geq 3)$$

with $t_0 = 0$, $t_1 = c$ and $t_2 = c^2 + 2c$.

Proof. We use the function $T(z)$ in Theorem 3.1 as it is. Let $T(z) = \sum_{n=0}^{\infty} t_n z^n$ ($|z| < 1$). Then

$$\begin{aligned}
&T(z) - (c+3)zT(z) - (c^2-2c-3)z^2T(z) - (c^3-c^2+c+1)z^3T(z) - cz(1-z) \\
&= t_0 + \big(t_1 - (c+3)t_0 - c\big)z + \big(t_2 - (c+3)t_1 - (c^2-2c-3)t_0 + c\big)z^2 \\
&\quad + \sum_{n=3}^{\infty}\big(t_n - (c+3)t_{n-1} - (c^2-2c-3)t_{n-2} - (c^3-c^2+c+1)t_{n-3}\big)z^n \\
&= 0.
\end{aligned}$$

Hence, $t_0 = 0$, $t_1 - (c+3)t_0 - c = 0$, $t_2 - (c+3)t_1 - (c^2-2c-3)t_0 + c = 0$ and $t_n - (c+3)t_{n-1} - (c^2-2c-3)t_{n-2} - (c^3-c^2+c+1)t_{n-3} = 0$ ($n \geq 3$), yielding

$$t_n = (c+3)t_{n-1} - (c^2-2c-3)t_{n-2} - (c^3-c^2+c+1)t_{n-3} \quad (n \geq 3)$$

with $t_0 = 0$, $t_1 = c$, and $t_2 = c^2 + 2c$. □

Theorem 3.3. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k F_k^{(3)} = s_n,$$

where the numbers s_n satisfy the recurrence relation

$$s_n = (d+3c)s_{n-1} + (d^2-2cd-3c^2)s_{n-2} + (d^3-cd^2+c^2d+c^3)s_{n-3} \quad (n \geq 3)$$

with $s_0 = 0$*,* $s_1 = d$ *and* $s_2 = d(d+2c)$*.*

Proof. Replace c and z by $c^{-1}d$ and cz, respectively in the function $T(z)$. Then

$$T(z) = \frac{dz(1-cz)}{1-(d+3c)z-(d^2-2cd-3c^2)z^2-(d^3-cd^2+c^2d+c^3)z^3}\,.$$

Let $T(z) = \sum_{n=0}^{\infty} s_n z^n$ $(|z|<1)$. Since

$$\begin{aligned}
&T(z)-(d+3c)zT(z)-(d^2-2cd-3c^2)z^2T(z)-(d^3-cd^2+c^2d+c^3)z^3T(z)-dz(1-cz)\\
&= s_0+\big(s_1-(d+3c)s_0-d\big)z+\big(s_2-(d+3c)s_1-(d^2-2cd-3c^2)s_0+cd\big)z^2\\
&\quad+\sum_{n=3}^{\infty}\big(s_n-(d+3c)s_{n-1}-(d^2-2cd-3c^2)s_{n-2}-(d^3-cd^2+c^2d+c^3)s_{n-3}\big)z^n\\
&= 0\,,
\end{aligned}$$

we have $s_0 = 0$, $s_1-(d+3c)s_0-d = 0$, $s_2-(d+3c)s_1-(d^2-2cd-3c^2)s_0+cd = 0$ and $s_n-(d+3c)s_{n-1}-(d^2-2cd-3c^2)s_{n-2}-(d^3-cd^2+c^2d+c^3)s_{n-3} = 0$ $(n\ge 3)$. □

4. Fibonacci s-Step Numbers

In general, for $s\ge 2$ let $F_n^{(s)}$ be Fibonacci s-step numbers defined by $F_n^{(s)} = F_{n-1}^{(s)}+F_{n-2}^{(s)}+\cdots+F_{n-s}^{(s)}$ $(n\ge s)$ with $F_0^{(s)}=0$, $F_1^{(s)}=F_2^{(s)}=1$, $F_3^{(s)}=2$, …, $F_{s-1}^{(s)}=2^{s-3}$.

Theorem 4.1. *The ordinary power series generating function* $F_s(z)$ *of* $\sum_{k=0}^{n}\binom{n}{k}c^kF_k^{(s)}$ *is given by*

$$F_s(z) = \frac{cz(1-z)^{s-2}}{1-\sum_{k=1}^{s}\left(\sum_{j=0}^{k-1}(-1)^j\binom{s-k+j}{j}c^{k-j}-(-1)^k\binom{s}{k}\right)z^k}\,.$$

The proof is similar in nature to that of Theorem 3.1 and details are available upon request.

Example 4.2. *Set* $s=4$ *in Theorem 4.1.* $F_n^{(4)}$ *are called Tetranacci numbers, or Fibonacci 4-step numbers, defined by* $F_n^{(4)} = F_{n-1}^{(4)}+F_{n-2}^{(4)}+F_{n-3}^{(4)}+F_{n-4}^{(4)}$ $(n\ge 4)$ *with* $F_0^{(4)}=0$*,* $F_1^{(4)}=F_2^{(4)}=1$ *and* $F_3^{(4)}=2$*. The ordinary power series generating function* $F_4(z)$ *of* $\sum_{k=0}^{n}\binom{n}{k}c^kF_k^{(4)}$ *is given by*

$$F_4(z) = \frac{cz(1-z)^2}{1-(c+4)z-(c^2-3c-6)z^2-(c^3-2c^2+3c+4)z^3-(c^4-c^3+c^2-c-1)z^4}.$$

By applying Theorem 4.1 we obtain the following two theorems.

Theorem 4.3. *For* $n\ge 0$ *we have*

$$\sum_{k=0}^{n}\binom{n}{k}c^kF_k^{(s)} = \rho_n\,,$$

where the numbers ρ_n *satisfy the recurrence relation*

$$\rho_n = \sum_{k=1}^{s}\left(\sum_{j=0}^{k-1}(-1)^j\binom{s-k+j}{j}c^{k-j}-(-1)^k\binom{s}{k}\right)\rho_{n-k}\quad (n\ge s)$$

with $\rho_0 = 0$ *and*

$$\rho_n = \frac{1}{4}\big((2c+1)^n + 2nc - 1\big) \quad (n = 1, 2, \dots, s-1).$$

Theorem 4.4. *For* $n \ge 0$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k F_k^{(s)} = \mu_n,$$

where the numbers μ_n *satisfy the recurrence relation*

$$\mu_n = \sum_{k=1}^{s} \left(\sum_{j=0}^{k-1} (-1)^j \binom{s-k+j}{j} d^{k-j} c^j - (-1)^k \binom{s}{k} c^k \right) \mu_{n-k} \quad (n \ge s)$$

with $\mu_0 = 0$ *and*

$$\mu_n = \frac{1}{4}\big((c+2d)^n + 2nc^{n-1}d - c^n\big) \quad (n = 1, 2, \dots, s-1). \tag{4.1}$$

Proof of Theorem 4.4. We shall prove Theorem 4.4. Theorem 4.3 can be similarly proven. Replace c and z by $c^{-1}d$ and cz, respectively in the function $F_s(z)$. We call the function $G(z)$ in place of $F_s(z)$; Namely,

$$G(z) = \frac{dz(1-cz)^{s-2}}{1 - \sum_{k=1}^{s} \xi_k z^k},$$

where

$$\xi_k = \sum_{j=0}^{k-1} (-1)^j \binom{s-k+j}{j} d^{k-j} c^j - (-1)^k \binom{s}{k} c^k.$$

Let $G(z) = \sum_{n=0}^{\infty} \mu_n z^n$ ($|z| < 1$). Since

$$\left(1 - \sum_{k=1}^{s} \xi_k z^k\right) G(z) - dz(1-cz)^{s-2} = 0,$$

we have $\mu_0 = 0$,

$$\mu_n - \sum_{k=1}^{n} \xi_k \mu_{n-k} + (-1)^n \binom{s-2}{n-1} dc^{n-1} = 0 \quad (n = 1, 2, \dots, s-1)$$

and

$$\mu_n - \sum_{k=1}^{s} \xi_k \mu_{s-k} = 0 \quad (n \ge s).$$

We shall prove (4.1) by induction. It is clear that $\mu_1 = d$. Assume that (4.1) holds for μ_1, μ_2, …, μ_{n-1}. Then

$$\begin{aligned}
\mu_n &= \sum_{k=1}^{n} \xi_k \mu_{n-k} - (-1)^n \binom{s-2}{n-1} dc^{n-1} \\
&= \frac{1}{4}\Bigg(\sum_{k=1}^{n-1} d(c+d)^{k-1}\big((c+d)^{n-k} - c^{n-k} + 2(n-k)dc^{n-k-1}\big) \\
&\quad + 4(-1)^{n-1} \frac{(-1)^{n-1} n!}{(n-1)!} dc^{n-1} \Bigg) \\
&= \frac{1}{4}\Bigg((c+2d)^n - (c+2d)(c+d)^{n-1} - c(c+d)^{n-1} + c^n \\
&\quad + 2nd\big((c+d)^{n-1} - c^{n-1}\big) + 2c(c+d)^{n-1} - 2c^n - 2(n-1)d(c+d)^{n-1} \Bigg) \\
&= \frac{1}{4}\big((c+2d)^n - c^n + 2ndc^{n-1}\big).
\end{aligned}$$

This completes the proof of Theorem 4.4. □

By setting $s = 4$ in Theorem 4.3 and Theorem 4.4, respectively, we have the following results about Tetranacci numbers $F_k^{(4)}$.

Example 4.5. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^k F_k^{(4)} = t_n,$$

where the numbers t_n satisfy the recurrence relation

$$\begin{aligned}
t_n = (c+4)t_{n-1} + (c^2 - 3c - 6)t_{n-2} + (c^3 - 2c^2 + 3c + 4)t_{n-3} \\
+ (c^4 - c^3 + c^2 - c - 1)t_{n-4} \quad (n \geq 4)
\end{aligned}$$

with $t_0 = 0$, $t_1 = c$, $t_2 = c(c+2)$ and $t_3 = c(2c^2 + 3c + 3)$.

Example 4.6. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k F_k^{(4)} = s_n,$$

where the numbers s_n satisfy the recurrence relation

$$\begin{aligned}
s_n = (d+4c)s_{n-1} + (d^2 - 3cd - 6c^2)s_{n-2} + (d^3 - 2cd^2 + 3c^2 d + 4c^3)s_{n-3} \\
+ (d^4 - cd^3 + c^2 d^2 - c^3 d - c^4)s_{n-4} \quad (n \geq 4)
\end{aligned}$$

with $s_0 = 0$, $s_1 = d$, $s_2 = d(d+2c)$ and $s_3 = d(2d^2 + 3cd + 3c^2)$.

In the method of ordinary generating functions, an integer n for $\binom{n}{k}$ is restricted to be nonnegative. However, identities mentioned in previous sections hold for negative n too. Notice that for $n = -r < 0$

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

For example, consider the identity in Theorem 2.3. Notice that for any integer n

$$\lambda_n = \frac{1}{\sqrt{a^2+4b}}(\beta^n - \bar{\beta}^n),$$

where

$$\beta = \frac{(ad+2c)+d\sqrt{a^2+4b}}{2} \quad \text{and} \quad \bar{\beta} = \frac{(ad+2c)-d\sqrt{a^2+4b}}{2}.$$

Since $d\Psi + c = \beta$, we have for any integer n

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{n}{k} c^{n-k} d^k \Psi^k &= c^n \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{d\Psi}{c}\right)^k \\ &= c^n \left(1 + \frac{d\Psi}{c}\right)^n \\ &= (d\Psi + c)^n = \beta^n. \end{aligned}$$

Similarly, by $d\bar{\Psi} + c = \bar{\beta}$, we have

$$\sum_{k=0}^{\infty} \binom{n}{k} c^{n-k} d^k \bar{\Psi}^k = \bar{\beta}^n.$$

Thus,

$$\sum_{k=0}^{\infty} \binom{n}{k} c^{n-k} d^k (\Psi^k - \bar{\Psi}^k) = \beta^n - \bar{\beta}^n.$$

Dividing both sides by $\sqrt{a^2+4b}$, we obtain the following result, which is a generalization of Theorem 2.3.

Theorem 4.7. *For any integer n,*

$$\sum_{k=0}^{\infty} \binom{n}{k} c^{n-k} d^k u_k = \lambda_n,$$

where the numbers λ_n $(n \geq 0)$ are defined in Theorem 2.3.

5. Future Research Problems

It would be interesting to investigate the ordinary power series generating functions about generalized s-step Fibonacci numbers $\hat{F}_n^{(s)}$, satisfying the recurrence relation $\hat{F}_n^{(s)} = a_1 \hat{F}_{n-1}^{(s)} + a_2 \hat{F}_{n-2}^{(s)} + \cdots + a_s \hat{F}_{n-s}^{(s)}$.

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