FIBONACCI CONTRACTIONS OF CONTINUED FRACTIONS

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Abstract. We evaluate two continued fractions whose elements contain Fibonacci numbers indexed by the Fibonacci and Lucas sequences. One of the results obtained is

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{F_{F_1}/F_{F_4}}{1} - \frac{F_{F_2}/F_{F_5}}{1} + \frac{F_{F_3}/F_{F_6}}{1} + \frac{F_{F_4}/F_{F_7}}{1} - \frac{F_{F_5}/F_{F_8}}{1} + \frac{F_{F_6}/F_{F_9}}{1} + \cdots.$$

Similar results with other rapidly growing sequences of subscripts are provided and associated summation theorems are also given. These results are shown to fit naturally in the context of a general transformation formula for arbitrary continued fractions due to Oskar Perron.

1. Two Continued Fractions

The purpose of this paper is to prove and provide context for the following two continued fraction identities which seem to have escaped notice.

Theorem 1.1.

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{F_{F_1}/F_{F_4}}{1} - \frac{F_{F_2}/F_{F_5}}{1} + \frac{F_{F_3}/F_{F_6}}{1} + \frac{F_{F_4}/F_{F_7}}{1} - \frac{F_{F_5}/F_{F_8}}{1} + \frac{F_{F_6}/F_{F_9}}{1} + \cdots, \tag{1.1}$$

and

$$\frac{1+\sqrt{5}}{3} = 1 + \frac{F_{L_1}/F_{L_4}}{1} - \frac{F_{L_2}/F_{L_5}}{1} + \frac{F_{L_3}/F_{L_6}}{1} + \frac{F_{L_4}/F_{L_7}}{1} - \frac{F_{L_5}/F_{L_8}}{1} + \frac{F_{L_6}/F_{L_9}}{1} + \cdots. \tag{1.2}$$

In the above theorem, $\{F_n\}$ and $\{L_n\}$ denote the Fibonacci and Lucas sequences, respectively with $F_0 = 0$ and $L_0 = 2$. Also, the signs of the numerator elements have period 3. Originally, the identity (1.1) was presented by the author during the Illinois Number Theory Conference in 1997 [1] as a special case of Perron's Theorem, Theorem 4.1, described in Section 4.

As a byproduct we also obtain the following summation identities.

Theorem 1.2.

$$\frac{\sqrt{5}-1}{2} = \sum_{i=1}^{\infty} \frac{(-1)^{F_{i+1}-1} F_{F_i}}{F_{F_{i+1}} F_{F_{i+2}}}, \tag{1.3}$$

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and

$$\frac{\sqrt{5}-1}{2} = \sum_{i=1}^{\infty} \frac{(-1)^{L_i - 1} F_{L_{i-1}}}{F_{L_i} F_{L_{i+1}}}. \tag{1.4}$$

These identities bear some resemblance to the Millin series [7, 10],

$$\frac{7-\sqrt{5}}{2} = \sum_{i=0}^{\infty} \frac{1}{F_{2^i}}. \tag{1.5}$$

In fact, we give a general identity that implies all three and uncover a continued fraction version of (1.5).

Besides giving new results, another purpose of the paper is to provide a unified approach to a number of formulas relating to Fibonacci numbers and related infinite processes. Indeed the fact that a number of beautiful series and continued fraction expansions, as well as Fibonacci number identities all result from the same principle seems to have escaped widespread notice. In the context of continued fractions, the same idea also explains the general continued fraction contraction formula of Oskar Perron [9].

After the proofs Section 4 provides a context for the results and gives some variants as well as generalizations. Section 2 provides basic facts about continued fractions.

2. Continued Fraction Facts

We employ the usual notation

$$b_0 + \frac{a_1}{b_1} \, + \, \frac{a_2}{b_2} \, + \, \frac{a_3}{b_3} \, + \, \cdots \, + \, \frac{a_n}{b_n} \tag{2.1}$$

for the finite continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots \, \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n}}}}}.$$

The non-negative integer n is called the *length* of (1.2). The numbers a_i and b_i are called its *numerator* and *denominator elements*, respectively. The notation

$$b_0 + \frac{a_1}{b_1} \, + \, \frac{a_2}{b_2} \, + \, \frac{a_3}{b_3} \, + \, \cdots \tag{2.2}$$

denotes the limit as $n \to \infty$ of (1.2). $-$ signs are sometimes employed instead of $+$ signs to indicate that the negative of the next numerator element is to be used. The rational function (2.1) is called the *nth classical approximant* of the infinite continued fraction (2.2).

Associated with the continued fraction (2.2) are two sequences P_n and Q_n called its *classical numerators* and *classical denominators*, respectively. They are defined by the following initial conditions and recurrences. $P_0 = b_0$, $Q_0 = 1$, $P_1 = b_0 b_1 + a_1$, $Q_1 = b_1$, and for $n \geq 2$ by the *fundamental recurrences*:

$$\begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2} \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}. \end{aligned} \tag{2.3}$$

Then it is well-known (see[6]) that the quotient P_n/Q_n is equal to the nth classical approximant of (2.2). These recurrences can be put in a convenient matrix form [8] as follows:

$$\prod_{i=1}^{n+1}\begin{pmatrix} b_{i-1} & a_i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_n & a_{n+1}P_{n-1} \\ Q_n & a_{n+1}Q_{n-1} \end{pmatrix}. \tag{2.4}$$

(This equation is easily seen to be equivalent to (2.3) by replacing the first n factors on the left-hand side of (2.4) with the right-hand side where n has been replaced with $n-1$.) Taking the determinant of both sides yields the *determinant formula*: $P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n+1}a_1a_2\cdots a_n$.

To tidy up the continued fractions in the paper, *equivalence transformations* are employed, see [6]. Let d_n for $n \geq 1$ be non-zero. Then the classical approximant (2.1) and

$$b_0 + \frac{d_1a_1}{d_1b_1} + \frac{d_1d_2a_2}{d_2b_2} + \frac{d_2d_3a_3}{d_3b_3} + \cdots \frac{d_{n-1}d_na_n}{d_nb_n} \tag{2.5}$$

are equal. Equation (2.5) is referred to as an equivalence transformation of the continued fraction (2.2). Note that equivalence transformations *do* change the classical numerators and denominators in a simple way, although these relations will not be needed here.

We also use the well-known connection with series. For $n \geq 1$,

$$\frac{P_n}{Q_n} = b_0 + \sum_{i=1}^{n}\frac{(-1)^{i-1}a_1a_2\cdots a_i}{Q_{i-1}Q_i}. \tag{2.6}$$

Finally, later in the paper we will use the *segments* of continued fractions. The segments of (2.2) are continued fractions having corresponding classical numerators $P_{v,\lambda}$ and classical denominators $Q_{v,\lambda}$. These are the classical numerators and denominators of the continued fraction:

$$\frac{P_{v,\lambda}}{Q_{v,\lambda}} = b_\lambda + \frac{a_{\lambda+1}}{b_{\lambda+1}} + \frac{a_{\lambda+2}}{b_{\lambda+2}} + \frac{a_{\lambda+3}}{b_{\lambda+3}} + \cdots + \frac{a_{\lambda+v}}{b_{\lambda+v}}. \tag{2.7}$$

The determinant formula for segments is then:

$$P_{n,\lambda}Q_{n-1,\lambda} - P_{n-1,\lambda}Q_{n,\lambda} = (-1)^{n+1}a_{\lambda+1}a_{\lambda+2}\cdots a_{\lambda+n}.$$

3. Proof of the Theorems

This section gives a brief self-contained proof of Theorems 1 and 2. The proof is based on the following lemma.

Lemma 3.1. *Let $k \in \mathbb{Z}$ and let g_v be any sequence of integers with $g_{v-1} \neq g_v$. Then,*

$$F_{k+g_{v+1}} = \frac{F_{g_{v+1}-g_{v-1}}}{F_{g_v-g_{v-1}}}F_{k+g_v} + \frac{(-1)^{g_v-g_{v-1}-1}F_{g_{v+1}-g_v}}{F_{g_v-g_{v-1}}}F_{k+g_{v-1}}. \tag{3.1}$$

The proof follows almost immediately from the bilinear index reduction formula of Johnson [3, 4]:

$$F_aF_b - F_cF_d = (-1)^r[F_{a-r}F_{b-r} - F_{c-r}F_{d-r}], \tag{3.2}$$

where $a, b, c, d, r \in \mathbb{Z}$ and $a + b = c + d$. Although (3.2) was proved as a solution to problem B-960 in [11], that proof is just a specialization of equation (3.32) of [2], which is given *without proof* in [2]. Johnson [4], however, gives an elegant and short derivation, which is included at the beginning of the proof below, since it illustrates the theme underlying the results of this paper.

Proof. Let

$$A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then it is well-known that for $n \in \mathbb{Z}$,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = A^n. \tag{3.3}$$

Using (3.3) with $n = a$, b, c, and d, where $a + b = c + d$, in the associativity identity $A^a A^b = A^c A^d$ and equating the $(2, 2)$ matrix elements on both sides yields after re-arrangement $F_a F_b - F_c F_d = (-1)[F_{a-1}F_{b-1} - F_{c-1}F_{d-1}]$. Equation (3.2) now follows by induction.

The lemma is obtained by setting $a = r = g_v - g_{v-1}$, $b = k + g_{v+1}$, $c = k + g_v$, and $d = g_{v+1} - g_{v-1}$ in (3.2), and then dividing through by $F_{g_v - g_{v-1}}$. □

Proof of Theorems 1.1 and 1.2. We first establish the following identities involving finite continued fractions of length v.

$$\frac{F_{F_{v+2}-1}}{F_{F_{v+2}}} = \frac{1}{1} + \frac{F_{F_2}/F_{F_1}}{F_{F_3}/F_{F_1}} + \frac{F_{F_3}/F_{F_2}}{F_{F_4}/F_{F_2}} - \cdots + \frac{(-1)^{F_{v-1}-1} F_{F_v}/F_{F_{v-1}}}{F_{F_{v+1}}/F_{F_{v-1}}}, \tag{3.4}$$

and

$$\frac{F_{L_{v+1}-1}}{F_{L_{v+1}}} = \frac{1}{2} - \frac{F_{L_1}/F_{L_0}}{F_{L_2}/F_{L_0}} + \frac{F_{L_2}/F_{L_1}}{F_{L_3}/F_{L_1}} + \cdots + \frac{(-1)^{L_{v-2}-1} F_{L_{v-1}}/F_{L_{v-2}}}{F_{L_v}/F_{L_{v-2}}}. \tag{3.5}$$

It is easy to check directly that (3.4) holds for $v = 2$ and $v = 3$ and also that (3.5) holds for $v = 1$ and $v = 2$. For $v > 3$ and $v > 2$, respectively, one only needs to check that the fundamental recurrences (2.3) holds for these continued fractions, where for (3.4), $P_v = F_{F_{v+2}-1}$, $Q_v = F_{F_{v+2}}$, $a_v = (-1)^{F_{v-1}-1} F_{F_v}/F_{F_{v-1}}$, and $b_v = F_{F_{v+1}}/F_{F_{v-1}}$; while for (3.5), $P_v = F_{L_{v+1}-1}$, $Q_n = F_{L_{v+1}}$, $a_v = (-1)^{L_{v-2}-1} F_{L_{v-1}}/F_{L_{v-2}}$, and $b_v = F_{L_v}/F_{L_{v-2}}$.

With these values the equalities (2.3) follow from (3.1); to get the recurrences for P_v and Q_v in the case of the continued fraction (3.4), let $g_v = F_{v+1}$. Then the recurrence for P_v is the $k = -1$ case of (3.1), while the recurrence for Q_v is the $k = 0$ case. Similarly, the recurrences for (3.5) are obtained from the special case $g_v = L_v$, and again $k = -1$ and $k = 0$. Thus the identities (3.4) and (3.5) are established.

Theorem 1.1 follows from (3.4) and (3.5) by letting $v \to \infty$, absorbing initial two and three terms of the continued fractions, respectively, into the quadratic irrational on the left-hand side, and applying equivalence transformations.

Theorem 1.2 follows immediately from (3.4) and (3.5) by using (2.6) and letting $v \to \infty$. □

It is also possible to absorb the initial segments of (3.4) and (3.5) into the left-hand side and so obtain a family of variants of Theorems 1.1 and 1.2.

4. Context, Generalizations, and Variants

The continued fractions in Theorem 1.1 can be viewed as arising from accelerating the regular continued fractions for the golden ratio, which has all elements equal to 1, via the general contraction theorem of Perron [9], Theorem 4.1 below. Perron's Theorem allows one to compute the continued fraction whose approximants are a proper subsequence of the approximants of a given continued fraction. Usually Perron's Theorem is only applied to relatively simple subsequences, such as the sequence of even approximants, or odd approximants, (which yield the *even* and *odd* parts of the original continued fraction, respectively [6, 9]) since one usually does not have "nice" formulas for arbitrary numerators and denominators of segments.

However, in the case of continued fractions with constant element sequences, the classical numerators and denominator sequences are solutions of second-order linear difference equations with constant coefficients, i.e. generalized Fibonacci-type sequences. Hence applying Perron's Theorem to these continued fractions yields new continued fractions with elements that are simple functions of Fibonacci-type sequences. The theorems of the previous section are a special case of this idea.

We present the contraction theorem from Perron [9] in the following form.

Theorem 4.1. *Let $\{n_i\}_{i\geq 0}$ be a strictly increasing sequence of integers with $n_0 \geq 0$. Then*

$$\frac{P_{n_i}}{Q_{n_i}} = \delta_0 + \frac{\gamma_1}{\delta_1} \underset{+}{} \frac{\gamma_2}{\delta_2} \underset{+}{} \cdots \underset{+}{} \frac{\gamma_i}{\delta_i}, \tag{4.1}$$

where $\delta_0 = P_{n_0}/Q_{n_0}$, $\delta_1 = Q_{n_1}$,

$$\gamma_1 = (-1)^{n_0} a_1 a_2 \cdots a_{n_0+1} \frac{Q_{n_1-n_0-1,n_0+1}}{Q_{n_0}},$$

$$\gamma_2 = (-1)^{n_1-n_0-1} a_{n_0+2} a_{n_0+3} \cdots a_{n_1+1} \frac{Q_{n_0} Q_{n_2-n_1-1,n_1+1}}{Q_{n_1-n_0-1,n_0+1}},$$

$$\delta_2 = \frac{Q_{n_2-n_0-1,n_0+1}}{Q_{n_1-n_0-1,n_0+1}},$$

and for $v > 2$,

$$\gamma_v = (-1)^{n_{v-1}-n_{v-2}-1} a_{n_{v-2}+2} a_{n_{v-2}+3} \cdots a_{n_{v-1}+1} \cdot \frac{Q_{n_v-n_{v-1}-1,n_{v-1}+1}}{Q_{n_{v-1}-n_{v-2}-1,n_{v-2}+1}},$$

and

$$\delta_v = \frac{Q_{n_v-n_{v-2}-1,n_{v-2}+1}}{Q_{n_{v-1}-n_{v-2}-1,n_{v-2}+1}}.$$

Moreover, for $i > 0$, P_{n_i} (resp. Q_{n_i}) is the ith classical numerator (resp. denominator) of the continued fraction in equation (4.1). This remains true for $i = 0$, if the 0th classical denominator on the right is taken to be Q_{n_0}, instead of 1.

Employing (2.6) immediately gives the following corollary.

Corollary 1. *Under the hypotheses of Theorem 4.1,*

$$\frac{P_{n_v}}{Q_{n_v}} = \frac{P_{n_0}}{Q_{n_0}} + \sum_{k=1}^{v} (-1)^{n_{k-1}} \frac{Q_{n_k-n_{k-1}-1,n_{k-1}+1}}{Q_{n_{k-1}} Q_{n_k}} a_1 a_2 \cdots a_{n_{k-1}+1}.$$

This corollary is equivalent to the case of Theorem 3.1 of [5] in which f is strictly increasing.

Remarks.

(1) The continued fraction in the contraction theorem given by Perron on page 12 of [9] is actually an equivalence transformation of the one above so that the continued fraction elements are polynomials instead of rational functions. However applying the equivalence transformations destroys the correspondence between the numerator and denominator of the fraction on the left-hand side of (4.1) with the classical numerators and denominators of the continued fraction on the right-hand side. Thus we have given the theorem in the form of its penultimate step of the derivation from page 11 of [9], where the numerator and denominator on the left-hand side are actually equal

to the classical numerators and denominators of the continued fraction on the right. This form is important for employing (2.6) to obtain Corollary 1. After (4.1), one can always polish the resulting continued fraction with an equivalence transformation to suit one's aesthetic taste.

(2) We outline the proof of this theorem and describe the role played by the associativity of the matrix product to illustrate the similarity to the proof in Section 3.

To find the elements of a continued fraction having a given set of classical numerators and denominators, one considers the system (2.3), and for each n, solves for a_n and b_n. By Cramer's rule, the resulting formulas are ratios of determinants. This result is known as Bernoulli's Theorem in the theory of continued fractions [6]. If we desire the numerators and denominators to be a subsequence of the numerators and denominators of a given continued fraction (denoted by P_{n_i} and Q_{n_i}), then the determinants that occur are of the form $P_{n_{i-1}}Q_{n_i} - P_{n_i}Q_{n_{i-1}}$ and $P_{n_{i-2}}Q_{n_i} - P_{n_i}Q_{n_{i-2}}$. But due to the fact that the sequences P and Q are themselves classical numerators and denominators, simplification is possible. The reason for the simplification is the associativity of the matrix product. Specifically,

$$\prod_{k=0}^{m+n} \begin{pmatrix} b_{\lambda+k} & a_{\lambda+k+1} \\ 1 & 0 \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} b_{\lambda+i} & a_{\lambda+i+1} \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{m} \begin{pmatrix} b_{\lambda+n+j} & a_{\lambda+n+j+1} \\ 1 & 0 \end{pmatrix},$$

which by (2.4) and (2.7) can be written as

$$\begin{pmatrix} P_{n+m,\,\lambda} & a_{\lambda+n+m+1}P_{n+m-1,\,\lambda} \\ Q_{n+m,\,\lambda} & a_{\lambda+n+m+1}Q_{n+m-1,\,\lambda} \end{pmatrix} =$$

$$\begin{pmatrix} P_{n,\,\lambda} & a_{\lambda+n+1}P_{n-1,\,\lambda} \\ Q_{n,\,\lambda} & a_{\lambda+n+1}Q_{n-1,\,\lambda} \end{pmatrix} \begin{pmatrix} P_{m-1,\,\lambda+n+1} & a_{\lambda+n+m+1}P_{m-2,\,\lambda+n+1} \\ Q_{m-1,\,\lambda+n+1} & a_{\lambda+n+m+1}Q_{m-2,\,\lambda+n+1} \end{pmatrix}.$$

Computing the first column in this matrix product (the second is redundant) gives the *classical numerator segment associativity relations* [8]:

$$P_{n+m,\,\lambda} = P_{n,\,\lambda}P_{m-1,\,\lambda+n+1} + a_{\lambda+n+1}P_{n-1,\,\lambda}Q_{m-1,\,\lambda+n+1},$$

$$Q_{n+m,\,\lambda} = Q_{n,\,\lambda}P_{m-1,\,\lambda+n+1} + a_{\lambda+n+1}Q_{n-1,\,\lambda}Q_{m-1,\,\lambda+n+1}.$$

Finally, multiplying the first of these equations by $Q_{n,\,\lambda}$ and the second by $P_{n,\,\lambda}$, taking the difference between the resulting equations, and employing the determinant formula for segments yields the *generalized segment determinant formula* [8]:

$$P_{n,\lambda}Q_{n+m,\,\lambda} - P_{n+m,\,\lambda}Q_{n,\,\lambda} = (-1)^{n+1}a_{\lambda+1}\cdots a_{\lambda+n+1}Q_{m-1,\,\lambda+n+1}.$$

Using this formula to simplify the determinants in Bernoulli formula gives Perron's contraction theorem.

Although Theorem 4.1 may appear a bit unwieldy, there are three cases in which it simplifies substantially.

First: if the differences $n_i - n_{i-1}$ and $n_i - n_{i-2}$ are bounded. Then one only has to deal with polynomials of bounded degree (in the variables a_i and b_j). For example, making the sequence n_i equal to the sequence of even numbers yields the "even part" of a continued fraction. Then the difference $n_i - n_{i-2}$ is equal to 4. After an equivalence transformation this gives rise to partial numerators in the contracted continued fraction which are of fourth degree in the variables a_i and b_j, see [6, 9]. Even and odd parts of continued fractions are frequently seen and are not considered here.

Second: if $a_n = a$ and $b_n = b$, for constants a and b, for all $n \in \mathbb{Z}^+$, then the segments of the continued fraction reduce to the classical numerators and classical denominators. In this situation it is easy to compute these sequences as they are solutions of linear homogeneous second order difference equations with constant coefficients.

Third: if the sequence n_i is of Fibonacci type (satisfying $n_i = n_{i-1} + n_{i-2}$), then the differences $n_i - n_{i-1}$ and $n_i - n_{i-2}$ will be the same sequence (shifted) and a simplification in the formula will occur.

Notice that the results from the first section of this paper arise in the confluence of the last two cases. This confluence explains the rather pleasant appearance of the resulting identities.

The nicest example of Theorem 4.1 in the second case is obtained when the theorem is applied to the continued fraction for the golden ratio. In this situation, $P_n = F_{n+2}$ and $Q_n = F_{n+1}$. Indeed $Q_{\nu,\lambda} = F_{\nu+1}$.

To simplify matters, it is helpful to let the sequence n_ν be the *sum* of a sequence of positive integers, so that the differences in the subscripts on the right-hand sides of Theorem 3 simplify. That is, let $n_0 = m_0 \geq 0$, and for $\nu \geq 1$ put $n_\nu = m_0 + m_1 + \cdots + m_\nu$, where $m_i \in \mathbb{Z}^+$ for $i \geq 1$. Then $n_\nu - n_{\nu-1} = m_\nu$, and $n_\nu - n_{\nu-2} = m_{\nu-1} + m_\nu$ for $\nu \geq 2$. Note that the increased complexity of the left-hand side is not a problem, since one knows the limit. Finally, one can apply an equivalence transformation to the right-hand side of (4.1) to simplify the continued fraction. Applying these substitutions to Theorem 4.1 and Corollary 1 yields the following corollary.

Corollary 2. *Let the sequences $\{m_i\}_{i\geq 0}$ and $\{n_i\}_{i\geq 0}$ be as in the last paragraph. Then for $i \geq 1$,*

$$\begin{aligned} &\frac{(-1)^{m_1-1}F_{m_0}F_{n_{i+1}-m_0-m_1}}{F_{n_{i+1}-m_0}} = \\ &\frac{(-1)^{m_1-1}F_{m_0}F_{m_2}}{F_{m_1+m_2}} + \frac{(-1)^{m_2-1}F_{m_1}F_{m_3}}{F_{m_2+m_3}} + \cdots + \frac{(-1)^{m_i-1}F_{m_{i-1}}F_{m_{i+1}}}{F_{m_i+m_{i+1}}}. \end{aligned} \tag{4.2}$$

Also

$$\frac{F_{n_i+2}}{F_{n_i+1}} = \frac{F_{n_0+2}}{F_{n_0+1}} + \sum_{k=1}^{i}(-1)^{n_{k-1}}\frac{F_{n_k-n_{k-1}}}{F_{n_{k-1}+1}F_{n_k+1}}. \tag{4.3}$$

Note that (4.3) is equivalent to the case of Corollary 3.3 of [5] in which g is strictly increasing.

We consider a few special cases. Setting $m_i = 2^i$ in (4.2) yields after routine simplification

$$\frac{F_{2^{i+2}-1}}{F_{2^{i+2}-2}} = F_{3\cdot 2^0} - \frac{F_{2^0}\cdot F_{2^2}}{F_{3\cdot 2^1}} - \frac{F_{2^1}\cdot F_{2^3}}{F_{3\cdot 2^2}} - \cdots \frac{F_{2^{i-1}}\cdot F_{2^{i+1}}}{F_{3\cdot 2^i}}. \tag{4.4}$$

Letting $i \to \infty$ in (4.4) gives

$$\frac{\sqrt{5}-1}{2} = F_{3\cdot 2^0} - \frac{F_{2^0}\cdot F_{2^2}}{F_{3\cdot 2^1}} - \frac{F_{2^1}\cdot F_{2^3}}{F_{3\cdot 2^2}} - \cdots. \tag{4.5}$$

For this choice of n_i, after letting $i \to \infty$, (4.3) simplifies to (1.5). This shows how (4.5) is the continued fraction manifestation of Milin's series. The cases $m_i = F_{i+1}$ and $m_i = L_{i+1}$ yield (1.3) and (1.4), respectively (and also (1.1) and (1.2)). Setting $m_i = i!$ and $n_i = (i+1)! - 1$

in (4.2) and (4.3), respectively and letting $i \to \infty$ yield, respectively,

$$\frac{\sqrt{5}-1}{2} = \frac{F_{0!} \cdot F_{2!}}{F_{3 \cdot 1!}} - \frac{F_{1!} \cdot F_{3!}}{F_{4 \cdot 2!}} - \frac{F_{2!} \cdot F_{4!}}{F_{5 \cdot 3!}} - \cdots, \tag{4.6}$$

and

$$\frac{5-\sqrt{5}}{2} = \frac{F_{1 \cdot 1!}}{F_{1!}F_{2!}} + \frac{F_{2 \cdot 2!}}{F_{2!}F_{3!}} + \frac{F_{3 \cdot 3!}}{F_{3!}F_{4!}} + \cdots. \tag{4.7}$$

Of course many of the generalizations and variants of (1.5) follow similarly.

Finally consider the third case discussed above. Let N_i satisfy $N_i = N_{i-1} + N_{i-2}$, for $i \geq 1$ and assume that $N_i > N_{i-1}$ for $i \geq 1$. (Note that $N_{-1} > 0$.) We have the following corollary.

Corollary 3 (Fibonacci Contraction Formula)**.** *For $i \geq 0$*

$$\frac{P_{N_{i+1}-N_0-1}}{Q_{N_{i+1}-N_0-1}} = \delta_0 + \frac{\gamma_1}{\delta_1} + \frac{\gamma_2}{\delta_2} + \cdots + \frac{\gamma_i}{\delta_i},$$

where $\delta_0 = P_{N_{-1}-1}/Q_{N_{-1}-1}$, $\delta_1 = Q_{N_1-1}$,

$$\gamma_1 = (-1)^{N_1-1} a_1 a_2 \cdots a_{N_{-1}} \frac{Q_{N_0,N_{-1}}}{Q_{N_{-1}-1}},$$

and for $v \geq 2$,

$$\gamma_v = (-1)^{N_{v-2}-1} a_{N_{v-1}-N_0+1} a_{N_{v-1}-N_0+2} \cdots a_{N_v-N_0} \cdot Q_{N_{v-3}-1,N_{v-1}-N_0} Q_{N_{v-1}-1,N_v-N_0},$$

and

$$\delta_v = Q_{N_v-1,N_{v-1}-N_0+1}.$$

Proof. This follows immediately from Theorem 4.1 by letting $n_i = N_{i+1} - N_0 - 1$ and simplifying using the recurrence for the sequence N_i. Note that this selection for the sequence n_i satisfies the required conditions. □

5. Conclusion

This paper shows that a number of results from the literature relating to continued fraction, Fibonacci numbers, and infinite series have their genesis in the associativity of the matrix product. The core of the proof of the general contraction theorem of Perron is an example of this idea.

The methods here do not seem to easily give evaluations for continued fractions involving elements of the forms L_{F_v} or L_{L_v}. It would be interesting to know if similar results involving these sequences can be obtained in any way.

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