SOME IDENTITIES VIA GEOMETRIC SERIES

MARTIN GRIFFITHS

Abstract. In this article we demonstrate how to obtain, via the manipulation of certain geometric series, a number of identities arising from a particular infinite family of linear, second-order, homogeneous, recurrence relations.

1. Introduction

From Binet's formula [1, 3, 4] we know that

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right),$$

where ϕ is the *golden ratio* given by

$$\phi = \frac{1+\sqrt{5}}{2}.$$

The Fibonacci numbers may thus be regarded as an 'almost-geometric' sequence with common ratio ϕ in the sense that

$$\lim_{n\to\infty}\left(-\frac{1}{\phi}\right)^n = 0.$$

The sum of the series

$$F_1 + F_2 + F_3 + \cdots + F_n$$

may therefore be approximated using the formula for the sum of the finite geometric progression [5]

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

given by

$$\frac{a\left(r^n - 1\right)}{r-1}, \tag{1.1}$$

where $a, r \in \mathbb{R}$, $n \in \mathbb{N}$ and $r \neq 1$.

We in fact use a related idea here to obtain *exact* expressions for certain sums of finite series, the terms of which arise from a particular infinite family of linear, second-order, homogeneous, recurrence relations. The recurrence relations considered in this paper are all of the form

$$u_n = ku_{n-1} \pm u_{n-2},$$

where $k \in \mathbb{N}$. As will be seen in due course, these allow us to determine sums having particularly simple forms.

2. Some Preliminaries

We provide here some results that will be used in later sections, and start by considering $u_n = ku_{n-1} + u_{n-2}$, which gives rise to specific instances of the generalized Fibonacci (or Horadam) sequence [2]. This recurrence relation has auxiliary equation $\lambda^2 - k\lambda - 1 = 0$, which in turn possesses the solutions

$$\alpha = \frac{k+\sqrt{k^2+4}}{2} \quad \text{and} \quad \overline{\alpha} = \frac{k-\sqrt{k^2+4}}{2} = -\frac{1}{\alpha}.$$

Therefore,

$$u_n = a\alpha^n + b\overline{\alpha}^n = a\alpha^n + b\left(-\frac{1}{\alpha}\right)^n$$

for some $a, b \in \mathbb{R}$. As will become clear, we are interested here in the cases $a = -b$ and $a = b$. The former gives rise to the following generalization of the Binet formula

$$U_n = \frac{1}{\sqrt{k^2+4}}\left(\alpha^n - \left(-\frac{1}{\alpha}\right)^n\right), \tag{2.1}$$

while the latter results in

$$V_n = \alpha^n + \left(-\frac{1}{\alpha}\right)^n. \tag{2.2}$$

Note that $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = k$.

Next we show, by induction, that

$$\alpha^n = \alpha U_n + U_{n-1}. \tag{2.3}$$

First, (2.3) certainly holds for $n = 1$. Now assume that it is true for some $n \geq 1$. Then, on utilizing both the inductive hypothesis and the recurrence relation for U_n, we obtain

$$\begin{aligned}
\alpha^{n+1} &= \alpha^2 U_n + \alpha U_{n-1} \\
&= \left(\frac{k+\sqrt{k^2+4}}{2}\right)^2 U_n + \left(\frac{k+\sqrt{k^2+4}}{2}\right) U_{n-1} \\
&= \frac{1}{2}\left(k^2 + 2 + k\sqrt{k^2+4}\right) U_n + \frac{1}{2}\left(k + \sqrt{k^2+4}\right) U_{n-1} \\
&= \frac{1}{2}\left(k + \sqrt{k^2+4}\right)(kU_n + U_{n-1}) + U_n \\
&= \frac{1}{2}\left(k + \sqrt{k^2+4}\right) U_{n+1} + U_n \\
&= \alpha U_{n+1} + U_n,
\end{aligned}$$

as required. In a similar manner, it may be shown that (2.3) is also true when α is replaced by $\overline{\alpha}$.

3. An Initial Result

In this section we find simple expressions for the sum of the following finite series:

$$U_0 + U_{2m} + U_{4m} + \cdots + U_{2rm}, \tag{3.1}$$

where $m, r \in \mathbb{N}$. This is a somewhat more straightforward matter than that of obtaining the corresponding sum of terms from the sequence $(V_n)_{n\geq 0}$, which will be considered in a later section. We deal first with the simplest case, in which m is even. We therefore set $m = 2p$ for some $p \in \mathbb{N}$. Let us now consider the following finite geometric series:

$$1 + \alpha^{4p} + \alpha^{8p} + \cdots + \alpha^{4rp} = \sum_{j=0}^{r} \alpha^{4jp}.$$

Using (1.1) and (2.1), we obtain

$$\begin{aligned}
\sum_{j=0}^{r} \alpha^{4jp} &= \frac{\alpha^{4p(r+1)} - 1}{\alpha^{4p} - 1} \\
&= \frac{\alpha^{2p(r+1)}\left(\alpha^{2p(r+1)} - \alpha^{-2p(r+1)}\right)}{\alpha^{2p}\left(\alpha^{2p} - \alpha^{-2p}\right)} \\
&= \alpha^{2pr}\left(\frac{\alpha^{2p(r+1)} - \left(-\frac{1}{\alpha}\right)^{2p(r+1)}}{\alpha^{2p} - \left(-\frac{1}{\alpha}\right)^{2p}}\right) \\
&= \alpha^{2pr}\left(\frac{\frac{1}{\sqrt{k^2+4}}\left(\alpha^{2p(r+1)} - \left(-\frac{1}{\alpha}\right)^{2p(r+1)}\right)}{\frac{1}{\sqrt{k^2+4}}\left(\alpha^{2p} - \left(-\frac{1}{\alpha}\right)^{2p}\right)}\right) \\
&= \alpha^{2pr}\frac{U_{2p(r+1)}}{U_{2p}}. \qquad (3.2)
\end{aligned}$$

Incidentally, the above makes it clear why the case $a = -b$ was chosen in Section 2.

Now, using (2.3), we may write (3.2) as

$$1 + \sum_{j=1}^{r} \left(\alpha U_{4jp} + U_{4jp-1}\right) = \left(\alpha U_{2pr} + U_{2pr-1}\right) \frac{U_{2p(r+1)}}{U_{2p}}.$$

Since α is irrational, it is the case that, for $a, b, c, d \in \mathbb{Q}$, $a\alpha + b = c\alpha + d$ if, and only if, $a = c$ and $b = d$. It follows from this that

$$\sum_{j=0}^{r} U_{4jp} = \frac{U_{2pr}U_{2p(r+1)}}{U_{2p}}$$

and

$$\sum_{j=1}^{r} U_{4jp-1} = \frac{U_{2pr-1}U_{2p(r+1)}}{U_{2p}} - 1, \qquad (3.3)$$

remembering that $U_0 = 0$.

4. Companion Series

Before obtaining more identities associated with $(U_n)_{n\geq 0}$, we need to consider the sequence $(V_n)_{n\geq 0}$, which, for a given $k \in \mathbb{N}$, may be regarded as a companion to $(U_n)_{n\geq 0}$. Not only do these two sequences share the same recurrence relation, but $(V_n)_{n\geq 0}$ also satisfies an identity corresponding to (2.3), as follows:

$$\alpha^n \sqrt{k^2 + 4} = \alpha V_n + V_{n-1}. \qquad (4.1)$$

It is easily verified that this is true for $n = 1$, noting, from the definition of this sequence in (2.2), that $V_0 = 2$ and $V_1 = k$. In order to show that (4.1) is true in general, induction may be utilized once more, and indeed interested readers might like to check the details.

5. Further Results

We are now in a position to be able to consider the sum of the series

$$U_0 + U_{2m} + U_{4m} + \cdots + U_{2rm},$$

where m is odd. The cases r odd and r even are dealt with separately. To this end, let $m = 2p-1$ and $r = 2q-1$. We start by looking at the following finite geometric series:

$$1 + \alpha^{2(2p-1)} + \alpha^{4(2p-1)} + \cdots + \alpha^{2(2q-1)(2p-1)} = \sum_{j=0}^{2q-1} \alpha^{2j(2p-1)}.$$

We have

$$\begin{aligned}
\sum_{j=0}^{2q-1} \alpha^{2j(2p-1)} &= \frac{\alpha^{4q(2p-1)} - 1}{\alpha^{2(2p-1)} - 1} \\
&= \frac{\alpha^{2q(2p-1)}\left(\alpha^{2q(2p-1)} - \alpha^{-2q(2p-1)}\right)}{\alpha^{2p-1}\left(\alpha^{2p-1} - \alpha^{-(2p-1)}\right)} \\
&= \alpha^{(2q-1)(2p-1)}\sqrt{k^2+4}\left(\frac{\frac{1}{\sqrt{k^2+4}}\left(\alpha^{2q(2p-1)} - \left(-\frac{1}{\alpha}\right)^{2q(2p-1)}\right)}{\alpha^{2p-1} + \left(-\frac{1}{\alpha}\right)^{2p-1}}\right) \\
&= \alpha^{(2q-1)(2p-1)}\sqrt{k^2+4}\left(\frac{U_{2q(2p-1)}}{V_{2p-1}}\right). \qquad (5.1)
\end{aligned}$$

Using (2.3) and (4.1), we may write (5.1) as

$$1 + \sum_{j=1}^{2q-1}\left(\alpha U_{2j(2p-1)} + U_{2j(2p-1)-1}\right) = \left(\alpha V_{(2q-1)(2p-1)} + V_{(2q-1)(2p-1)-1}\right)\frac{U_{2q(2q-1)}}{V_{2p-1}},$$

which, because α is irrational, leads to the results

$$\sum_{j=0}^{2q-1} U_{2j(2p-1)} = \frac{U_{2q(2p-1)}V_{(2q-1)(2p-1)}}{V_{2p-1}}$$

and

$$\sum_{j=1}^{2q-1} U_{2j(2p-1)-1} = \frac{U_{2q(2p-1)}V_{(2q-1)(2p-1)-1}}{V_{2p-1}} - 1.$$

Finally, in this section, we obtain the sum of the series (3.1) where m is odd and r is even; say $m = 2p-1$ and $r = 2q$. We consider the following finite geometric series:

$$1 + \alpha^{2(2p-1)} + \alpha^{4(2p-1)} + \cdots + \alpha^{4q(2p-1)} = \sum_{j=0}^{2q} \alpha^{2j(2p-1)}.$$

We then have

$$\begin{aligned}\sum_{j=0}^{2q}\alpha^{2j(2p-1)} &= \frac{\alpha^{2(2q+1)(2p-1)}-1}{\alpha^{2(2p-1)}-1}\\ &= \frac{\alpha^{(2q+1)(2p-1)}\left(\alpha^{(2q+1)(2p-1)}-\alpha^{-(2q+1)(2p-1)}\right)}{\alpha^{2p-1}\left(\alpha^{2p-1}-\alpha^{-(2p-1)}\right)}\\ &= \alpha^{2q(2p-1)}\left(\frac{\alpha^{(2q+1)(2p-1)}+\left(-\frac{1}{\alpha}\right)^{(2q+1)(2p-1)}}{\alpha^{2p-1}+\left(-\frac{1}{\alpha}\right)^{2p-1}}\right)\\ &= \alpha^{2q(2p-1)}\frac{V_{(2q+1)(2p-1)}}{V_{2p-1}}. \qquad (5.2)\end{aligned}$$

Using (2.3), we may write (5.2) as

$$1+\sum_{j=1}^{2q}\left(\alpha U_{2j(2p-1)}+U_{2j(2p-1)-1}\right)=\left(\alpha U_{2q(2p-1)}+U_{2q(2p-1)-1}\right)\frac{V_{(2q+1)(2p-1)}}{V_{2p-1}},$$

from which it follows that

$$\sum_{j=0}^{2q}U_{2j(2p-1)}=\frac{U_{2q(2p-1)}V_{(2q+1)(2p-1)}}{V_{2p-1}}$$

and

$$\sum_{j=1}^{2q}U_{2j(2p-1)-1}=\frac{U_{2q(2p-1)-1}V_{(2q+1)(2p-1)}}{V_{2p-1}}-1.$$

6. Sums Involving V_n

We now go on to obtain formulas for the following sum

$$V_0+V_{2m}+V_{4m}+\cdots+V_{2rm}.$$

Suppose first that $m=2p$ for some $p\in\mathbb{N}$. This time we consider

$$\sqrt{k^2+4}\left(1+\alpha^{4p}+\alpha^{8p}+\cdots+\alpha^{4rp}\right)=\sqrt{k^2+4}\left(\sum_{j=0}^{r}\alpha^{4jp}\right).$$

Using (1.1), we obtain, via similar manipulations to those used in obtaining (3.2),

$$\sqrt{k^2+4}\left(\sum_{j=0}^{r}\alpha^{4jp}\right)=\alpha^{2pr}\sqrt{k^2+4}\left(\frac{U_{2p(r+1)}}{U_{2p}}\right),$$

which, using (4.1), may be rewritten as

$$\sqrt{k^2+4}+\sum_{j=1}^{r}\left(\alpha V_{4jp}+V_{4jp-1}\right)=\left(\alpha V_{2pr}+V_{2pr-1}\right)\frac{U_{2p(r+1)}}{U_{2p}}.$$

Since $\sqrt{k^2+4}=2\alpha-k$, this in turn gives

$$2\alpha+\sum_{j=1}^{r}\left(\alpha V_{4jp}+V_{4jp-1}\right)=k+\left(\alpha V_{2pr}+V_{2pr-1}\right)\frac{U_{2p(r+1)}}{U_{2p}}. \qquad (6.1)$$

Finally then, on using (6.1) and remembering that $V_0 = 2$, we obtain the following results:

$$\sum_{j=0}^{r} V_{4jp} = \frac{V_{2pr}U_{2p(r+1)}}{U_{2p}}$$

and

$$\sum_{j=1}^{r} V_{4jp-1} = \frac{V_{2pr-1}U_{2p(r+1)}}{U_{2p}} + k.$$

Next, let $m = 2p-1$ and $r = 2q-1$. We start by considering the following finite geometric series:

$$\sqrt{k^2+4}\left(1 + \alpha^{2(2p-1)} + \alpha^{4(2p-1)} + \cdots + \alpha^{2(2q-1)(2p-1)}\right) = \sqrt{k^2+4}\left(\sum_{j=0}^{2q-1} \alpha^{2j(2p-1)}\right).$$

From (5.1), we have

$$\sqrt{k^2+4}\left(\sum_{j=0}^{2q-1} \alpha^{2j(2p-1)}\right) = \alpha^{(2q-1)(2p-1)}\left(k^2+4\right)\left(\frac{U_{2q(2p-1)}}{V_{2p-1}}\right).$$

This result, in conjunction with (2.3) and (4.1), gives

$$\sqrt{k^2+4} + \sum_{j=1}^{2q-1}\left(\alpha V_{2j(2p-1)} + V_{2j(2p-1)-1}\right)$$
$$= \left(k^2+4\right)\left(\alpha U_{(2q-1)(2p-1)} + U_{(2q-1)(2p-1)-1}\right)\frac{U_{2q(2p-1)}}{V_{2p-1}}.$$

Using $\sqrt{k^2+4} = 2\alpha - k$ and $V_0 = 2$ once more, we have

$$\sum_{j=0}^{2q-1} V_{2j(2p-1)} = \left(k^2+4\right)\frac{U_{2q(2p-1)}U_{(2q-1)(2p-1)}}{V_{2p-1}}$$

and

$$\sum_{j=1}^{2q-1} V_{2j(2p-1)-1} = \left(k^2+4\right)\frac{U_{2q(2p-1)}U_{(2q-1)(2p-1)-1}}{V_{2p-1}} + k.$$

With $m = 2p-1$ and $r = 2q$, we may use (5.2) to obtain

$$\sqrt{k^2+4}\left(\sum_{j=0}^{2q} \alpha^{2j(2p-1)}\right) = \alpha^{2q(2p-1)}\sqrt{k^2+4}\left(\frac{V_{(2q+1)(2p-1)}}{V_{2p-1}}\right).$$

Then (4.1) gives

$$\sqrt{k^2+4} + \sum_{j=1}^{2q}\left(\alpha V_{2j(2p-1)} + V_{2j(2p-1)-1}\right) = \left(\alpha V_{2q(2p-1)} + V_{2q(2p-1)-1}\right)\frac{V_{(2q+1)(2p-1)}}{V_{2p-1}},$$

from which it follows that

$$\sum_{j=0}^{2q} V_{2j(2p-1)} = \frac{V_{2q(2p-1)}V_{(2q+1)(2p-1)}}{V_{2p-1}}$$

and

$$\sum_{j=1}^{2q} V_{2j(2p-1)-1} = \frac{V_{2q(2p-1)-1}V_{(2q+1)(2p-1)}}{V_{2p-1}} + k.$$

7. The Alternative Recurrence Relation

We now briefly consider the corresponding situation for the recurrence relation

$$u_n = ku_{n-1} - u_{n-2},$$

where k is an integer such that $k \geq 3$. The auxiliary equation in this case is $\lambda^2 - k\lambda + 1 = 0$, which has the solutions

$$\beta = \frac{k + \sqrt{k^2-4}}{2} \quad \text{and} \quad \overline{\beta} = \frac{k - \sqrt{k^2-4}}{2} = \frac{1}{\beta}.$$

Therefore,

$$u_n = a\beta^n + b\overline{\beta}^n = a\beta^n + \frac{b}{\beta^n}$$

for some $a, b \in \mathbb{R}$. As before, we are interested here in the cases $a = -b$ and $a = b$. The former gives rise to the following series:

$$X_n = \frac{1}{\sqrt{k^2-4}} \left(\beta^n - \frac{1}{\beta^n} \right),$$

while the latter results in

$$Y_n = \beta^n + \frac{1}{\beta^n}.$$

Note that $X_0 = 0$, $X_1 = 1$, $Y_0 = 2$ and $Y_1 = k$. Furthermore, it is straightforward to show that

$$\beta^n = \beta X_n - X_{n-1}, \tag{7.1}$$

and

$$\beta^n \sqrt{k^2-4} = \beta Y_n - Y_{n-1}.$$

The form of β and $\overline{\beta}$ means that, unlike the situation for the sum (3.1), it is not necessary to split the results into cases. The sum of the series

$$X_0 + X_{2m} + X_{4m} + \cdots + X_{2rm}$$

is in fact given by

$$\sum_{j=0}^{r} X_{2jm} = \frac{X_{rm}X_{m(r+1)}}{X_m}.$$

We also have

$$\sum_{j=1}^{r} X_{2jm-1} = \frac{X_{rm-1}X_{m(r+1)}}{X_m} + 1,$$

noting that, because of the form of (7.1), the sign of the final term is different to that in (3.3).

Similarly

$$\sum_{j=0}^{r} Y_{2jm} = \frac{Y_{rm}X_{m(r+1)}}{X_m}$$

and

$$\sum_{j=1}^{r} Y_{2jm-1} = \frac{Y_{rm-1}X_{m(r+1)}}{X_m} - k.$$

Again here, interested readers might like to verify the results in this section for themselves.

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Department of Mathematics, Christ's College, Christchurch 8013, New Zealand
E-mail address: `mgriffiths@christscollege.com`