FUSION, FISSION, AND FACTORS

CLARK KIMBERLING

Abstract. Operations called fusion and fission are applied to sequences of polynomials and to infinite matrices. Special cases involving Fibonacci polynomials of the second kind are considered, with attention to Fibonacci self-fusion and self-fission matrices, factorizations of terms in these matrices, and factorizations of associated polynomials.

1. Introduction

We begin with an example and then generalize.

Example 1.1.

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 & 9 \end{pmatrix}$$

Interpreting rows as polynomials and matrix product as an operation $\odot$, we write

$$(2x^2+x+1) \odot \begin{pmatrix} x^3+x^2+2x+3 \\ x^2+x+2 \\ x+1 \end{pmatrix} = 2x^3+3x^2+6x+9, \tag{1}$$

and note that this polynomial factors using Fibonacci numbers:

$$(2x+3)(x^2+3). \tag{2}$$

Now suppose that

$$p = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0 \tag{3}$$

is a polynomial and that Q is a sequence of polynomials:

$$q_k(x) = q_{k,0}x^k + q_{k,1}x^{k-1} + \cdots + q_{k,k-1}x + q_{k,k}, \tag{4}$$

for $k = 0, 1, 2, \ldots$. The Q*-upstep of* p is defined by

$$u(p) = p_n q_{n+1}(x) + p_{n-1}q_n(x) + \cdots + p_0 q_1(x).$$

Note that $q_0(x)$ does not appear. Next let $P = (p_n(x))$ and $Q = (q_n(x))$ be sequences of polynomials, where p_n and q_n have degree n. The *fusion of* P *by* Q, denoted by $P \odot Q$, is the sequence $V = (v_n(x))$ given by $v_0(x) = 1$ and $v_{n+1}(x) = u(p_n(x))$. As suggested by Example 1.1, we may regard P and Q as numerical matrices and $\odot$ as matrix multiplication, so that row $n+1$ of $P \odot Q$, for $n \geq 0$, is given by the matrix product $P(n)\widehat{Q}(n)$, where

$$P(n) = \begin{pmatrix} p_{n,n} & p_{n,n-1} & \cdots & p_{n,1} & p_{n,0} \end{pmatrix}$$

and $\widehat{Q}(n)$ is the $(n+1) \times (n+2)$ matrix

$$\begin{pmatrix} q_{n+1,0} & q_{n+1,1} & \cdots & q_{n+1,n} & q_{n+1,n+1} \\ 0 & q_{n,0} & \cdots & q_{n,n-1} & q_{n,n} \\ 0 & 0 & \cdots & q_{n-1,n-2} & q_{n-1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & q_{2,1} & q_{2,2} \\ 0 & 0 & \cdots & q_{1,0} & q_{1,1} \end{pmatrix}. \tag{5}$$

Let p and $q_k(x)$ be as in (3) and (4). The *Q-downstep of p* is defined for $n > 0$ by

$$d(p) = p_n q_{n-1}(x) + p_{n-1} q_{n-2}(x) + \cdots + p_1 q_0(x),$$

where p_0 does not appear. As before, suppose that $P = (p_n(x))$ and $Q = (q_n(x))$ are sequences of polynomials, where p_n and q_n have degree n. The *fission of P by Q*, denoted by $P \circledast Q$, is the sequence $W = (w_n(x))$ of polynomials given by $w_0(x) = 1$ and $w_{n+1}(x) = d(p_{n+1}(x))$. We may regard $\circledast$ as an operation on matrices P and Q. In this case, row n of $P \circledast Q$, for $n > 0$, is given by the matrix product $\widetilde{P}(n+1)\widetilde{Q}(n)$, where

$$\widetilde{P}(n+1) = \begin{pmatrix} p_{n+1,n+1} & p_{n+1,n} & \cdots & p_{n+1,2} & p_{n+1,1} \end{pmatrix}$$

and $\widetilde{Q}(n)$ is the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} q_{n,0} & q_{n,1} & \cdots & q_{n,n-1} & q_{n,n} \\ 0 & q_{n-1,0} & \cdots & q_{n-1,n-2} & q_{n-1,n-1} \\ 0 & 0 & \cdots & q_{n-2,n-3} & q_{n-2,n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & q_{1,0} & q_{1,1} \\ 0 & 0 & \cdots & 0 & q_{0,0} \end{pmatrix}. \tag{6}$$

Note that for $n > 0$ the fission polynomial $w_n(x)$ has degree $n-1$, whereas the fusion polynomial $v_n(x)$ has degree n.

Example 1.2. *In order to compare fission and fusion, consider the equation*

$$\begin{pmatrix} 5 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 8 & 15 & 24 \end{pmatrix},$$

recast as

$$(5x^4 + 3x^3 + 2x^2 + x) \circledast \begin{pmatrix} x^3 + x^2 + 2x + 3 \\ x^2 + x + 2 \\ x + 1 \\ 1 \end{pmatrix} = 5x^3 + 8x^2 + 15x + 24,$$

which factors using Fibonacci numbers:

$$(5x + 8)(x^2 + 3). \tag{7}$$

These expressions are analogous to those in (1) and (2).

2. Fibonacci Matrices: Definitions

The Introduction indicates that fusion and fission can be studied both as polynomial sequences and as matrices. With regard to polynomials, the focus in later sections will be on recurrences, factoring, and roots, whereas for matrices, we shall be interested in certain products and inverses. In this section, notation will be established for certain fundamental matrices. We begin with the (infinite) *upper triangular Fibonacci matrix*, U, in which row n consists of $n-1$ zeros followed by the Fibonacci sequence, $1, 1, 2, 3, 5, 8, \ldots$. Let U_n denote the nth principal submatrix of U; viz, U_4 occurs in Example 1.1. The *lower triangular Fibonacci matrix*, L, is the transpose of U, and L_n is the nth principal submatrix of L. The *Fibonacci self-fusion matrix* is the product $M = LU$; e.g., the 7th principal submatrix of M is

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 \\ 1 & 2 & 3 & 5 & 8 & 13 & 21 \\ 2 & 3 & 6 & 9 & 15 & 24 & 39 \\ 3 & 5 & 9 & 15 & 24 & 39 & 63 \\ 5 & 8 & 15 & 24 & 40 & 64 & 104 \\ 8 & 13 & 24 & 39 & 64 & 104 & 168 \\ 13 & 21 & 39 & 63 & 104 & 168 & 273 \end{pmatrix}. \tag{8}$$

Note that the result in Example 1.1 is included in (8) as the initial 4-tuple in row 3. More generally, the polynomials $v_n(x)$, defined by the upstep operation in Section 1, are given by the first $n+1$ terms of row n of M. In [3] and [4], the nth principal submatrix of M is called the symmetric Fibonacci matrix; various factorizations are given and properties are proved.

The *modified lower triangular Fibonacci matrix*, $\widetilde{L}$, is obtained by deleting the first row and the principal diagonal of L. The *Fibonacci self-fission matrix* is the product $\widetilde{M} = \widetilde{L}U$, with 7th principal submatrix

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 \\ 2 & 3 & 5 & 8 & 13 & 21 & 34 \\ 3 & 5 & 9 & 14 & 23 & 37 & 60 \\ 5 & 8 & 15 & 24 & 39 & 63 & 102 \\ 8 & 13 & 24 & 39 & 64 & 103 & 167 \\ 13 & 21 & 39 & 63 & 104 & 168 & 272 \\ 21 & 34 & 63 & 102 & 168 & 272 & 441 \end{pmatrix}, \tag{9}$$

which includes the result in Example 1.2 as the initial 4-tuple in row 4. More generally, the polynomials $w_n(x)$, defined by the downstep operation in Section 1, are given by the first n terms of row n of $\widetilde{M}$.

3. Fibonacci Matrices: Properties

Let $m(n,k)$ denote the general term of the Fibonacci self-fusion matrix, M, so that $m(n,k) =$ (row n of L) $\cdot$ (column k of U), given by

$$m(n,k) = \begin{cases} \sum_{i=1}^{n} F_{n+1-i}F_{k+1-i} & \text{if } n \le k \\ m(k,n) & \text{if } n > k. \end{cases} \tag{10}$$

Lemma 3.1. *Eventually, each row of M satisfies the Fibonacci recurrence; specifically, if $h \ge 2$, then $m(n, n+h) = m(n, n+h-1) + m(n, n+h-2)$.*

Proof.

$$\begin{aligned} m(n,n+h-1)+m(n,n+h-2) &= \sum_{i=1}^{n} F_{n+1-i}F_{n+h-i} + \sum_{i=1}^{n} F_{n+1-i}F_{n+h-i-1} \\ &= \sum_{i=1}^{n} F_{n+1-i}(F_{n+h-i} + F_{n+h-i-1}) \\ &= \sum_{i=1}^{n} F_{n+1-i}F_{n+h+1-i} \\ &= m(n,n+h). \end{aligned}$$

□

In row n of M, the terms up to $m(n,n+2)$ do not satisfy the recurrence in Lemma 3.1. Instead,

$$m(n,n) = \sum_{i=1}^{n} F_{n+1-i}^2 = F_nF_{n+1}, \tag{11}$$

a well-known identity, and for $m(n,n+1)$ we have the following lemma.

Lemma 3.2.

$$m(n,n+1) = \begin{cases} F_{n+1}^2 & \text{if } n \text{ is odd} \\ F_nF_{n+2} & \text{if } n \text{ is even.} \end{cases} \tag{12}$$

Proof. The identity clearly holds for $n = 1$ and $n = 2$. As an induction hypothesis, assume that (12) holds when n is replaced by an arbitrary $k \geq 2$. Then, if k is even,

$$\begin{aligned} m(k,k+1) &= F_kF_{k+1} + F_{k-1}F_k + \cdots + F_1F_2 \\ &= F_k(F_k + F_{k-1}) + F_{k-1}F_k + \cdots + F_1F_2 \\ &= F_kF_{k+1} + F_k^2 \\ &= F_kF_{k+2}. \end{aligned}$$

If k is odd and $k \geq 3$, then

$$\begin{aligned} m(k,k+1) &= F_kF_{k+1} + F_{k-1}F_k + \cdots + F_1F_2 \\ &= F_k(F_k + F_{k-1}) + F_{k-1}F_k + \cdots + F_1F_2 \\ &= F_kF_{k+1} + F_{k-1}F_{k+1} \\ &= F_{k+1}^2. \end{aligned}$$

□

Theorem 3.3. *Every term of the Fibonacci self-fusion matrix M is a product of two Fibonacci numbers:*

$$m(n,k) = \begin{cases} F_nF_{k+1} & \text{if } k \text{ is even} \\ F_{n+1}F_k & \text{if } k \text{ is odd.} \end{cases} \tag{13}$$

Proof. First, suppose that $n \leq k$. Trivially (13) holds for $n \in \{1,2\}$, and (13) holds for $k = n \geq 3$ by (11) and for $k = n+1$ by Lemma 3.2. Lemmas 3.1 and 3.2 and induction then imply that (13) holds for $m(n,k)$ for $n \leq k$ and $k \geq 2$. Finally, for $n > k$, (13) holds by the symmetry property in (10).

Turning now to the Fibonacci self-fission matrix $\widetilde{M}$, we have $\widetilde{m}(n,k)$ =(row n of $\widetilde{L}$)·(column k of U), so that

$$\widetilde{m}(n,k) = \begin{cases} \sum_{i=1}^{n} F_{n+2-i}F_{k+1-i} & \text{if } n \le k \\ \sum_{i=1}^{k} F_{n+2-i}F_{k+1-i} & \text{if } n > k. \end{cases}$$

□

Theorem 3.4. *The terms of the Fibonacci self-fission matrix $\widetilde{M}$ are represented by those of M as follows:*

$$\widetilde{m}(n,k) = \begin{cases} m(n+1,k) - F_{k-n} & \text{if } n < k \\ m(n+1,k) & \text{if } n \ge k. \end{cases}$$

Proof. First, suppose that $n < k$. Then by (10),

$$\begin{aligned} m(n+1,k) - F_{k-n} &= F_{n+1}F_k + F_nF_{k-1} + \cdots \\ &\quad + F_2F_{k-(n+1)+2} + F_1F_{k-(n+1)+1} - F_{k-n} \\ &= F_{n+1}F_k + F_nF_{k-1} + \cdots + F_2F_{k-n+1} \\ &= \widetilde{m}(n,k). \end{aligned}$$

Next, suppose that $n \ge k$. Then

$$\begin{aligned} \widetilde{m}(n,k) &= \sum_{i=1}^{k} F_{n+2-i}F_{k+1-i} \\ &= m(k,n+1) \text{ by (10)} \\ &= m(n+1,k) \text{ by (10)}. \end{aligned}$$

□

Corollary 3.5. *Eventually each row of $\widetilde{M}$ satisfies the Fibonacci recurrence; specifically, if $h \ge 1$, then $\widetilde{m}(n,n+h) = \widetilde{m}(n,n+h-1) + \widetilde{m}(n,n+h-2)$.*

Proof. This follows immediately from Lemma 3.1 and Theorem 3.4. □

4. Fibonacci Polynomials of the 2nd Kind

The Fibonacci polynomials $g_n(x)$ of the 2nd kind are defined [2] as partial sums of the generating function of the Fibonacci numbers:

$$g_n(x) = 1 + x + 2x^2 + \cdots + F_{n+1}x^n.$$

Let $f_n(x) = x^n g_n(x^{-1})$, so that the sequence $(f_n(x))$ is given by $f_0(x) = 1$ and $f_n(x) = xf_{n-1}(x) + F_{n+1}$ for $n > 0$. These *reversed Fibonacci polynomials of the 2nd kind* serve as a basis for some striking applications of the fission and fusion operators, $\odot$ and $\circledast$, defined in Section 1. In particular, we wish to account for the sort of factorization seen in (2) and (7).

In this section, the polynomials p and $q(x)$ in (3) and (4) are taken to be $f_n(x)$. The resulting polynomials $v_n(x)$ are then the Fibonacci *self-fusion polynomials*, and $w_n(x)$, the

Fibonacci self-fission polynomials.

Table 1. Polynomials		
Fibonacci self-fusion polynomials v_n and self-fission polynomials w_n		
n	$v_n(x)$	$w_n(x)$
0	1	1
1	$x+1$	1
2	x^2+2x+3	$2x+3$
3	$2x^3+3x^2+6x+9$	$3x^2+5x+9$
4	$3x^4+5x^3+9x^2+15x+24$	$5x^3+8x^2+15x+24$

Factorization properties of the polynomials $v_n(x)$ and $w_n(x)$ are given by the next two theorems.

Theorem 4.1. *The Fibonacci self-fusion polynomials $v_n(x)$ are given by two cases, according as n is odd or even. If $k \geq 1$, then*

$$v_{2k+1}(x) = (xF_{2k+1} + F_{2k+2})(F_2x^{2k} + F_4x^{2k-2} + \cdots + F_{2k}x^2 + F_{2k+2})$$
$$v_{2k}(x) - F_{2k}F_{2k+2} = x(xF_{2k} + F_{2k+1})(F_2x^{2k-2} + F_4x^{2k-4} + \cdots + F_{2k}).$$

Proof. Let $X_n = (x^n, x^{n-1}, \ldots, x, 1, 0, 0, 0, \ldots)$, and suppose that $k \geq 1$. Then

$$\begin{aligned}
v_{2k+1}(x) &= (\text{row } 2k+1 \text{ of } M) \cdot X_{2k+1} \\
&= (F_{2k+1}, F_{2k+2}, 3F_{2k+1}, 3F_{2k+2}, \ldots, F_{2k+2}F_{2k+1}, F_{2k+2}F_{2k+2}) \cdot X_{2k+1} \\
&= x^{2k}(xF_{2k+1} + F_{2k+2}) + 3x^{2k-2}(xF_{2k+1} + F_{2k+2}) + \cdots \\
&\quad + F_{2k+2}x^0(xF_{2k+1} + F_{2k+2}) \\
&= (xF_{2k+1} + F_{2k+2})(F_2x^{2k} + F_4x^{2k-2} + \cdots + F_{2k}x^2 + F_{2k+2}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
v_{2k}(x) &= (\text{row } 2k \text{ of } M) \cdot X_{2k} \\
&= (F_{2k}, F_{2k+1}, 3F_{2k+2}, 3F_{2k+3}, \ldots, F_{2k}^2, F_{2k}F_{2k+1}, F_{2k}F_{2k+2}) \cdot X_{2k} \\
&= x^{2k-1}(xF_{2k} + F_{2k+1}) + 3x^{2k-3}(xF_{2k} + F_{2k+1}) + \cdots \\
&\quad + F_{2k}x(xF_{2k} + F_{2k+1}) + F_{2k}F_{2k+2} \\
&= (xF_{2k} + F_{2k+1})(F_2x^{2k-1} + F_4x^{2k-3} + \cdots + F_{2k}x) + F_{2k}F_{2k+2} \\
&= x(xF_{2k} + F_{2k+1})(F_2x^{2k-2} + F_4x^{2k-4} + \cdots + F_{2k}) + F_{2k}F_{2k+2}.
\end{aligned}$$

□

Theorem 4.2. *The Fibonacci self-fission polynomials $w_n(x)$ are given by two cases, according as n is even or odd. If $k \geq 1$, then*

$$w_{2k}(x) = (xF_{2k+1} + F_{2k+2})(F_2x^{2k-2} + F_4x^{2k-4} + \cdots + F_{2k})$$
$$w_{2k+1}(x) - F_{2k+2}^2 = x(xF_{2k+2} + F_{2k+3})(F_2x^{2k-2} + F_4x^{2k-4} + \cdots + F_{2k}).$$

Proof. A proof follows the method for Theorem 4.1 and is omitted. □

Theorems 4.1 and 4.2 can be summarized in terms of the reversed polynomials $f_n(x)$ of the 2nd kind:

$$\begin{aligned} v_{2k+1}(x) &= (xF_{2k+1} + F_{2k+2})(f_{2k+1}(x) - f_{2k+1}(-x))/(2x) \\ v_{2k}(x) &= (xF_{2k} + F_{2k+1})(f_{2k-1}(x) - f_{2k-1}(-x))/2 + F_{2k}F_{2k+2} \\ w_{2k}(x) &= (xF_{2k+1} + F_{2k+2})(f_{2k}(x) - f_{2k}(-x))/(2x) \\ w_{2k+1}(x) &= (xF_{2k+2} + F_{2k+3})(f_{2k}(x) - f_{2k}(-x))/2 + F_{2k+2}^2. \end{aligned}$$

These results in Theorems 4.1 and 4.2 generalize as follows. First, for $r \geq 0$, define $v_{r,0}(x) = 1$ and $w_{r,0}(x) = 1$, and recalling the upstep and downstep operations in Section 1, define

$$v_{r,n+1}(x) = u(f_{n+r}(x)) \quad \text{and} \quad w_{r,n+1}(x) = d(f_{n+r+1}(x)).$$

Let L_r be the matrix obtained from L by deleting the first r rows, and let M_r be the matrix obtained by deleting the first r rows of M, so that $L_rU = M_r$. The methods used above for the case $r = 0$ then apply. Beginning with the fusion polynomials, $v_{r,n}(x)$ is read from row n of M_r, which is identical to row $n + r$ of M:

$$v_{r,n}(x) = m(n + r, 1)x^n + m(n + r, 2)x^{n-1} + \cdots + m(n + r, n - 1),$$

so that Theorem A applies: if n is odd and ≥ 3, then

$$v_{r,n}(x) = (xF_{n+r} + F_{n+r+1})(F_2x^{n-1} + F_4x^{n-3} + \cdots + F_{n+1}),$$

and if n is even and $n \geq 4$, then

$$v_{r,n}(x) = x(xF_{n+r} + F_{n+r+1})(F_2x^{n-2} + F_4x^{n-4} + \cdots + F_n) + F_{n+r}F_{n+2}.$$

Now, analogously, to generalize Theorem 4.2, let $\widetilde{L}_r$ be the matrix obtained from $\widetilde{L}$ by deleting the first r rows, and let $\widetilde{M}_r$ be the matrix obtained by deleting the first r rows of $\widetilde{M}$, so that $\widetilde{L}_rU = \widetilde{M}_r$. Then the fission polynomial, $w_{r,n}(x)$ is read from row n of $\widetilde{M}_r$, which is identical to row $n + r$ of $\widetilde{M}$:

$$w_{r,n}(x) = \widetilde{m}(n + r, 1)x^{n-1} + \widetilde{m}(n + r, 2)x^{n-2} + \cdots + \widetilde{m}(n + r, n),$$

so that Theorems 3.3 and 3.4 apply: if n is even and ≥ 4, then

$$w_{r,n}(x) = (xF_{n+r+1} + F_{n+r+2})(F_2x^{n-2} + F_4x^{n-4} + \cdots + F_n),$$

and if n is odd and $n \geq 5$, then

$$w_{r,n}(x) = x(xF_{n+r+1} + F_{n+r+2})(F_2x^{n-3} + F_4x^{n-5} + \cdots + F_{n-1}) + F_{n+r+1}F_{n+1}.$$

5. Concluding Remarks

The Online Encyclopedia of Integer Sequences [5] includes several sequence-representations of matrices mentioned in this paper. Each entry includes a `Mathematica` program that can be used to generate many more terms than have been shown above.

[A202451,] upper triangular Fibonacci matrix, U

[A202452,] lower triangular Fibonacci matrix, L

[A202453,] Fibonacci self-fusion matrix, M

[A202502,] modified lower triangular Fibonacci matrix, $\widetilde{L}$

[A202503,] Fibonacci self-fission array, $\widetilde{M}$

[A193722,] definition of fusion

[A193842,] definition of fission

[A202605,] interlacing of roots associated with the Fibonacci self-fusion matrix, M

The final sequence in the list, A202605, illustrates an interesting theorem [1] about interlacing roots. Since the Fibonacci self-fusion matrix, M, is symmetric, the characteristic roots of the successive principal submatrices of M are all real and are interlaced. Specifically, if $h_n(x) = (x - r_{n,1})(x - r_{n,2}) \cdots (x - r_{n,n})$ is the nth such polynomial, then

$$r_{n+1,1} < r_{n,1} < r_{n+1,2} < r_{n,2} < \cdots < r_{n+1,n} < r_{n,n} < r_{n+1,n+1}.$$

Approximations for the roots of $h_1(x)$ to $h_5(x)$ are shown here:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0.38 & & 2.62 & & & \\
 & & 0.28 & & 0.43 & & 8.29 & & \\
 & 0.26 & & 0.30 & & 0.56 & & 22.89 & \\
0.24 & & 0.27 & & 0.42 & & 0.60 & & 62.48
\end{array}$$

References

[1] S.-G. Hwang, *Cauchy's interlace theorem for eigenvalues of Hermitian matrices*, American Mathematical Monthly, **111** (2004), 157–159.

[2] C. Kimberling, *Partial sums of generating functions as polynomial sequences*, The Fibonacci Quarterly, **48.4** (2010), 327–334.

[3] G.-Y. Lee, J.-S. Kim, and S.-G. Lee, *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices*, The Fibonacci Quarterly, **40.3** (2002), 203–211.

[4] G.-Y. Lee and J.-S. Kim, *The linear algebra of the generalized Fibonacci matrices*, The Fibonacci Quarterly, **41.5** (2003), 451–465.

[5] OEIS Foundation Inc. (2013), The Online Encyclopedia of Integer Sequences, `http://oeis.org/`.

MSC2010: 11B39, 11C08, 11C20

Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, IN 47722

E-mail address: `ck6@evansville.edu`