A NOTE ON THE MODES OF THE POISSON DISTRIBUTION OF ORDER k

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Abstract. It is shown that the Poisson distribution of order k (≥ 1) with parameter λ (> 0) has a unique mode $m_{k,\lambda} = 0$ if $0 < \lambda < 2/(k(k+1))$. In addition, $m_{2,\lambda} = 0$ if $0 < \lambda \leq -1+\sqrt{3}$ and $m_{2,\lambda} = 2$ if $-1 + \sqrt{3} \leq \lambda < 1$.

1. Introduction

The Poisson distribution of order $k\,(\geq 1)$ with parameter $\lambda\,(> 0)$ has probability mass function (pmf)

$$f_k(x; \lambda) = e^{-k\lambda} \sum \frac{\lambda^{x_1+\cdots+x_k}}{x_1! \cdots x_k!}, \quad \text{for} \quad x = 0, 1, 2, \ldots, \tag{1.1}$$

where the summation is taken over all k-tuples of non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$ [1–9]. It was obtained in [9] as a limit of the negative binomial distribution of order k, and was named so because it reduces to the Poisson distribution with pmf $f_1(x; \lambda) = e^{-\lambda}\lambda^x/x!$ for $k = 1$.

Denote by $m_{k,\lambda}$ the mode(s) of $f_k(x; \lambda)$, i.e. the value(s) of x for which $f_k(x; \lambda)$ attains its maximum. It is well-known that $f_1(x; \lambda)$ has a unique mode $m_{1,\lambda} = \lfloor \lambda \rfloor$ if $\lambda \notin \mathbb{N}$, and two modes $m_{1,\lambda} = \lambda$ and $\lambda - 1$ if $\lambda \in \mathbb{N}$. It was established recently [3] that

$$\lfloor \mu_{k,\lambda} \rfloor - k(k + 1)/2 + 1 - \delta_{k,1} \leq m_{k,\lambda} \leq \lfloor \mu_{k,\lambda} \rfloor, \text{ for } \lambda > 0, k \geq 1, \tag{1.2}$$

and

$$m_{k,\lambda} = \mu_{k,\lambda} - \lfloor k/2 \rfloor, \text{ for } \lambda \in \mathbb{N}, 2 \leq k \leq 5, \tag{1.3}$$

where $\mu_{k,\lambda} = \lambda k(k + 1)/2$ is the mean of the Poisson distribution of order k [6], $\delta_{k,1}$ is the Kronecker delta, and $\lfloor u \rfloor$ denotes the greatest integer not exceeding $u \in \mathbb{R}$.

We presently show, as a consequence of (1.2), that $m_{k,\lambda} = 0$ if $0 < \lambda < 2/(k(k + 1))$ and $k \geq 1$ (see Proposition 2.1). We also show that $m_{2,\lambda} = 0$ if $0 < \lambda \leq -1 + \sqrt{3}$ and $m_{2,\lambda} = 2$ if $-1 + \sqrt{3} \leq \lambda < 1$ (see Proposition 2.2).

2. Main Results

In this section we state and prove the following propositions.

Proposition 2.1. *For any integer $k \geq 1$ and $0 < \lambda < 2/(k(k + 1))$, the Poisson distribution of order k has a unique mode $m_{k,\lambda} = 0$.*

Proposition 2.2. *The Poisson distribution of order 2 has a unique mode $m_{2,\lambda} = 0$ if $0 < \lambda < -1 + \sqrt{3}$. It has two modes $m_{2,\lambda} = 0$ and 2 if $\lambda = -1 + \sqrt{3}$, and it has a unique mode $m_{2,\lambda} = 2$ if $-1 + \sqrt{3} < \lambda < 1$.*

Proof of Proposition 2.1. We have

$$\begin{aligned} 0 &\le m_{k,\lambda} \text{ for } k \ge 1, \lambda > 0, \text{ by the definition of } m_{k,\lambda}, \\ &\le \lfloor \mu_{k,\lambda} \rfloor = \lfloor \lambda k(k+1)/2 \rfloor, \text{ by (1.2)}, \\ &= 0, \text{ since } 0 < \lambda k(k+1)/2 < 1 \text{ by the assumption.} \end{aligned}$$

For $k = 1$, the condition $0 < \lambda < 2/(k(k+1))$ is obviously necessary and sufficient for Proposition 2.1 to hold true. For $k = 2$, however, it is not necessary because of Proposition 2.2. □

Proof of Proposition 2.2. The definition of $m_{2,\lambda}$ and (1.2) imply

$$0 \le m_{2,\lambda} \le \lfloor \mu_{2,\lambda} \rfloor = \lfloor 3\lambda \rfloor \le 2 \text{ for } 0 < \lambda < 1.$$

Next, using (1.1) or recurrence (2.3) of [3], for $\lambda > 0$ we get

$$f_2(0;\lambda) = e^{-2\lambda}, \; f_2(1;\lambda) = \lambda e^{-2\lambda}, \text{ and } f_2(2;\lambda) = (\lambda + \frac{\lambda^2}{2})e^{-2\lambda}. \tag{2.1}$$

It follows that

$$f_2(0;\lambda) > f_2(1;\lambda) \text{ for } 0 < \lambda < 1, \text{ and } f_2(1;\lambda) < f_2(2;\lambda) \text{ for } \lambda > 0.$$

Therefore, in order to obtain $m_{2,\lambda}$ for $0 < \lambda < 1$, it suffices to compare $f_2(0;\lambda)$ and $f_2(2;\lambda)$ for $0 < \lambda < 1$. By means of (2.1),

$$f_2(2;\lambda) \le f_2(0;\lambda) \text{ if and only if } \lambda + \frac{\lambda^2}{2} \le 1 \text{ if and only if } 0 < \lambda \le -1 + \sqrt{3},$$

and

$$f_2(2;\lambda) \ge f_2(0;\lambda) \text{ if and only if } \lambda + \frac{\lambda^2}{2} \ge 1 \text{ if and only if } -1 + \sqrt{3} \le \lambda < 1,$$

which complete the proof of the proposition. □

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