ON A GENERALIZED PELL EQUATION AND A CHARACTERIZATION OF THE FIBONACCI AND LUCAS NUMBERS

RUSSELL EULER AND JAWAD SADEK

Abstract. A general method to solve the Pell equation $x^2 - dy^2 = a^2$ is given under certain conditions on a and d. As a special case, our method gives a different technique than the continued fractions technique used by C. T. Long and J. H. Jordan to characterize the Fibonacci and Lucas numbers as solutions to $x^2 - 5y^2 = \pm 4$.

1. Introduction

Consider the Pell equation

$$x^2 - dy^2 = a^2, \tag{1.1}$$

where $a^2 + d = b^2$ for some positive integer b. We will give a general method to solve (1.1). In [1], it was shown that (L_{2n+2}, F_{2n+2}) and (L_{2n-1}, F_{2n-1}) give all the solutions to the Pell equations $x^2 - 5y^2 = 4$, and $x^2 - 5y^2 = -4$, respectively. This was done using continued fractions. Our general solution to (1.1) leads to a different method to show that "unusual characterization of the Fibonacci and Lucas number" discussed in [1].

2. The Solution to the Pell Equation

Let

$$\begin{aligned} x_{n+1} &= \frac{bx_n + dy_n}{a} \\ y_{n+1} &= \frac{x_n + by_n}{a} \end{aligned} \tag{2.1}$$

with the smallest positive solution $(x_1, y_1) = (b, 1)$. We prove that the system (2.1) generates a family of solutions to (1.1).

First, we show if (x_n, y_n) is a solution to (1.1), then (x_{n+1}, y_{n+1}) is also a solution. In fact,

$$\begin{aligned} x_{n+1}^2 - dy_{n+1}^2 &= \frac{b^2x_n^2 + 2bdx_ny_n + d^2y_n^2}{a^2} - d\frac{x_n^2 + 2bx_ny_n + b^2y_n^2}{a^2} \\ &= \frac{a^2x_n^2 - a^2dy_n^2}{a^2} \\ &= x_n^2 - dy_n^2 \\ &= a^2. \end{aligned}$$

Now we show that the system (2.1), under certain conditions, gives all the solutions to (1.1). Assume for a contradiction that there exists a solution (u, v) such that

$$x_n + y_n\sqrt{d} < u + v\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d}. \tag{2.2}$$

Multiplying (2.2) by $x_n - y_n\sqrt{d}$ leads to

$$a^2 < (u+v\sqrt{d})(x_n - y_n\sqrt{d}) < \left(\frac{bx_n + dy_n}{a} + \frac{x_n + by_n}{a}\sqrt{d}\right)(x_n - y_n\sqrt{d}). \tag{2.3}$$

The rightmost expression in (2.3) reduces to

$$\begin{aligned}
&\frac{1}{a}\Big(bx_n^2 - bx_ny_n\sqrt{d} + dx_ny_n - d\sqrt{d}y_n^2 + bx_ny_n\sqrt{d} - bdy_n^2 + x_n^2\sqrt{d} - dx_ny_n\Big)\\
&\qquad = \frac{1}{a}\left[b(x_n^2 - dy_n^2) + (x_n^2 - dy_n^2)\sqrt{d}\right]\\
&\qquad = \frac{1}{a}(ba^2 + a^2\sqrt{d})\\
&\qquad = a(b+\sqrt{d}).
\end{aligned}$$

Thus,

$$a^2 < (u+v\sqrt{d})(x_n - y_n\sqrt{d}) < a(b+\sqrt{d}). \tag{2.4}$$

The middle term in (2.4) can be written as

$$ux_n - dvy_n + (vx_n - uy_n)\sqrt{d} = r + s\sqrt{d}.$$

Now under the condition that $\frac{r}{a}$ and $\frac{s}{a}$ are integers, it follows from dividing (2.4) by a that

$$a < \frac{r}{a} + \frac{s}{a}\sqrt{d} < b + \sqrt{d} \tag{2.5}$$

and so $(\frac{r}{a}, \frac{s}{a})$ is a solution to (1.1). In fact,

$$\begin{aligned}
\left(\frac{r}{a}\right)^2 - d\left(\frac{s}{a}\right)^2 &= \left(\frac{ux_n - dvy_n}{a}\right)^2 - d\left(\frac{vx_n - uy_n}{a}\right)^2\\
&= \frac{u^2x_n^2 - 2duvx_ny_n + d^2v^2y_n^2 - dv^2x_n^2 + 2duvx_ny_n - du^2y_n^2}{a^2}\\
&= \frac{(x_n^2 - dy_n^2)u^2 - dv^2(x_n^2 - dy_n^2)}{a^2}\\
&= \frac{(u^2 - dv^2)(x_n^2 - dy_n^2)}{a^2}\\
&= a^2.
\end{aligned}$$

Now we show that $\left(\frac{r}{a}, \frac{s}{a}\right)$ is a positive solution. Since $a^2 < r+s\sqrt{d}$ and $(r+s\sqrt{d})(r-s\sqrt{d}) = a^4$, $0 < r - s\sqrt{d} < a^2$. It follows that

$$2r = r + s\sqrt{d} + r - s\sqrt{d} > a^2 + 0 > 0$$

and

$$2s\sqrt{d} = r + s\sqrt{d} - (r - s\sqrt{d}) > a^2 - a^2 = 0.$$

We have shown that if there was a positive solution (u, v) between (x_n, y_n) and (x_{n+1}, y_{n+1}), then there would be a positive solution $\left(\frac{r}{a}, \frac{s}{a}\right)$ such that $\frac{r}{a} + \frac{s}{a}\sqrt{d} < b + \sqrt{d}$. This is a contradiction because $(b, 1)$ is the smallest positive solution to (1.1).

3. The Solution in Closed Form

Let $d = 5$ and $a = 2$. Then $d + a^2 = 3^2$ and so $b = 3$. It is simple to see that $x^2 - 5y^2 = 4$ implies that x_n, y_n, u, and v, defined above, have the same parity (for any positive integer n) and so r and s are even. Thus $\frac{r}{2}$ and $\frac{s}{2}$ are integers. Now we use standard linear algebra techniques to find the solution to (1.1) in closed form. In fact, the recurrence relation in (2.1) may be written as $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ where $A = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} \sqrt{5} \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -\sqrt{5} \\ 1 \end{bmatrix}$. Thus we can write

$$\begin{aligned} A &= \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{5}} & \frac{1}{2} \\ \frac{-1}{2\sqrt{5}} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

It follows that

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A^n \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{3+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{3-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{5}} & \frac{1}{2} \\ \frac{-1}{2\sqrt{5}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right] + \frac{\sqrt{5}}{2}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] \\ \frac{3}{2\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] + \frac{1}{2}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right] \end{bmatrix}. \end{aligned} \tag{3.1}$$

Using the facts $\beta^2 = \frac{3-\sqrt{5}}{2}$, $\alpha^2 = \frac{3+\sqrt{5}}{2}$, $L_n = \alpha^n + \beta^n$, and $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, the matrix in (3.1) reduces to $\begin{bmatrix} \alpha^{2n+2} + \beta^{2n+2} \\ \frac{1}{\sqrt{5}}\left(\alpha^{2n+2} - \beta^{2n+2}\right) \end{bmatrix} = \begin{bmatrix} L_{2n+2} \\ F_{2n+2} \end{bmatrix}$. We have the "unusual characterization of both Fibonacci and Lucas numbers" that had been shown in [1].

Now consider the Pell equation

$$x^2 - 5y^2 = -4. \tag{3.2}$$

The smallest positive solution is $(1, 1)$. Following the same arguments and techniques used in the solution to (2.1), it can be shown that all the solutions to (3.2) are given by the recurrence relation

$$\begin{aligned} x_{n+1} &= \frac{3x_n + 5y_n}{2} \\ y_{n+1} &= \frac{x_n + 3y_n}{2} \end{aligned} \tag{3.3}$$

with the initial solution $(x_1, y_1) = (1, 1)$. The only difference is we multiply the inequality in (2.2) by $y_n\sqrt{5} - x_n$ instead of $x_n - y_n\sqrt{5}$. Also the contradiction will still be the existence of a positive solution to (1.1) that is smaller than $(3, 1)$ and not a positive solution to (3.2) that is smaller than $(1, 1)$. Finally, the closed form of the solution to (3.3) is given by $x = L_{2n-1}$ and $y = F_{2n-1}$, where $n \geq 1$. The proof is identical to the case of equation (1.1).

References

[1] C. T. Long and J. H. Jordan, *A limited arithmetic on simple continued fractions*, The Fibonacci Quarterly, **5.2** (1967), 113–128.

MSC2010: 11B39, 11B99

Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, MO, 64468
E-mail address: reuler@nwmissouri.edu

Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, MO, 64468
E-mail address: jawads@nwmissouri.edu