ON APPROXIMATING EULER'S CONSTANT

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ABSTRACT. The aim of this paper is to improve the result obtained by Hirschhorn in 2011 about the inequalities for the Euler-Mascheroni constant.

1. Introduction

The sequence defined by

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n,\ n \geq 1,$$

is convergent to a limit denoted $\gamma = 0,5772\ldots$ now known as the Euler-Mascheroni constant. Several estimates for $\gamma_n - \gamma$ have been given in the literature, for example:

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)},\ n \geq 2, \quad [5]$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n},\ n \geq 1, \quad [7]$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n},\ n \geq 1, \quad [3,4]$$

$$\frac{1}{2n + \frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3}},\ n \geq 1, \quad [6]$$

$$\frac{1}{2n + \frac{2\gamma - 1}{1 - \gamma}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3}},\ n \geq 1. \quad [1,6]$$

Using the elementary technique of the Maclaurin series for $\ln(1+x)$, Hirschhorn [2] obtained in 2011 the following estimate:

$$\frac{1}{2n + \frac{1}{3} + \frac{1}{18n}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{1}{32n}},\ n \geq 2.$$

In this paper we obtain a better estimate for the right inequality and, for the left inequality we remark that $\frac{1}{18}$ is the best constant using an elementary sequence method.

2. The Main Result

Theorem 2.1. *(i) For every $a \geq \frac{1}{18}$ we have*

$$\frac{1}{2n + \frac{1}{3} + \frac{a}{n}} < \gamma_n - \gamma \text{ for all } n \geq 1.$$

(ii) For every $0 < a < \frac{1}{18}$ there exists $n_a \in N$ such that

$$\gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{a}{n}}, \text{ for all } n \geq n_a.$$

Consequently, $a = \frac{1}{18}$ is the best constant for the inequality of (i).

Proof. We consider the sequence

$$a_n = \gamma_n - \gamma - \frac{1}{2n + \frac{1}{3} + \frac{a}{n}}, \text{ for } a > 0,$$

and we show that $a_n > 0$ for all $n \geq 1$, if $a \geq \frac{1}{18}$, and, for $0 < a < \frac{1}{18}$ there exists $n_a \in N$ such that $a_n < 0$ for all $n \geq n_a$.
(i) Since (a_n) converges to zero we prove that (a_n) is strictly decreasing. Then we should look at $a_{n+1} - a_n = f(n)$, where

$$f(n) = \frac{1}{n+1} - \ln(n+1) + \ln n - \frac{1}{2n + \frac{7}{3} + \frac{a}{n+1}} + \frac{1}{2n + \frac{1}{3} + \frac{a}{n}}.$$

The derivative of function f is equal to

$$f'(n) = \frac{P(n)}{n(n+1)^2(6n^2+n+3a)^2(6n^2+13n+3a+7)^2},$$

where

$$\begin{aligned} P(n) = {} & 216(18a-1)n^6 + 312(45a-2)n^5 + (2916a^2 + 20322a - 551)n^4 \\ & + (7020a^2 + 14961a - 94)n^3 + (648a^3 + 6318a^2 + 5526a + 49)n^2 \\ & + (756a^3 + 2430a^2 + 735a)n + 81a^4 + 378a^3 + 441a^2. \end{aligned}$$

If $a = \frac{1}{18}$ then

$$P(n) = 156n^5 + 578n^4 + \frac{4553}{6}n^3 + \frac{6761}{18}n^2 + \frac{94211}{1944}n + \frac{1849}{1296} > 0,$$

for all $n \geq 1$, and then f is strictly increasing. We have $f(\infty) = 0$ and then it follows that $f(n) < 0$ for all $n \geq 1$, so that $(a_n)_{n\geq 1}$ is strictly decreasing. Since (a_n) converges to zero it follows that $a_n > 0$ for all $n \geq 1$, so that

$$\frac{1}{2n + \frac{1}{3} + \frac{1}{18n}} < \gamma_n - \gamma, \text{ for all } n \geq 1.$$

If $a > \frac{1}{18}$ then $\frac{1}{2n + \frac{1}{3} + \frac{a}{n}} < \frac{1}{2n + \frac{1}{3} + \frac{1}{18n}}$ and so

$$\frac{1}{2n + \frac{1}{3} + \frac{a}{n}} < \gamma_n - \gamma, \text{ for all } n \geq 1.$$

(ii) If $a < \frac{1}{18}$ then there exists $n_a \in N$ such that $P(n) < 0$ for all $n \geq n_a$ and then f is strictly decreasing on $[n_a, \infty)$. Since $f(\infty) = 0$ it follows that $f(n) > 0$ for all $n \geq n_a$, so that $(a_n)_{n\geq n_a}$

is strictly increasing. The sequence (a_n) converges to zero and then it follows that $a_n < 0$ for all $n \geq n_a$, so that

$$\gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{a}{n}} \text{ for all } n \geq n_a.$$

□

Now we find the constant n_a in some particular cases. For example, if $a = \frac{1}{31} \in (\frac{1}{32}, \frac{1}{18})$, then

$$P(n) = -\frac{2808}{31}n^6 - \frac{5304}{31}n^5 + \frac{103387}{961}n^4 + \frac{380477}{961}n^3 + \frac{6966751}{29791}n^2 + \frac{782421}{29791}n + \frac{435600}{923521} < 0,$$

for all $n \geq 2$, and so

$$\frac{1}{2n + \frac{1}{3} + \frac{1}{18n}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{1}{31n}}, \text{ for all } n \geq 2.$$

If $a = \frac{1}{19} \in (\frac{1}{32}, \frac{1}{18})$, then

$$P(n) = -\frac{216}{19}n^6 + \frac{2184}{19}n^5 + \frac{190123}{361}n^4 + \frac{251027}{361}n^3 + \frac{2451667}{6859}n^2 + \frac{312261}{6859}n + \frac{166464}{130321} < 0,$$

for all $n \geq 13$ and so

$$\frac{1}{2n + \frac{1}{3} + \frac{1}{18n}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{1}{19n}}, \text{ for all } n \geq 13.$$

Let us remark that a direct calculus shows that these inequalities hold and for $n \in \{9, 10, 11, 12\}$, and then

$$\frac{1}{2n + \frac{1}{3} + \frac{1}{18n}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3} + \frac{1}{19n}}, \text{ for all } n \geq 9.$$

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