THREE SERIES FOR THE GENERALIZED GOLDEN MEAN

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Abstract. As is well-known, the ratio of adjacent Fibonacci numbers tends to $\phi = (1 + \sqrt{5})/2$, and the ratio of adjacent Tribonacci numbers (where each term is the sum of the three preceding numbers) tends to the real root η of $X^3 - X^2 - X - 1 = 0$. Letting α_n denote the corresponding ratio for the generalized Fibonacci numbers, where each term is the sum of the n preceding, we obtain rapidly converging series for α_n, $1/\alpha_n$, and $1/(2 - \alpha_n)$.

1. Introduction

The Fibonacci numbers are defined by the recurrence

$$F_i = F_{i-1} + F_{i-2}$$

with initial values $F_0 = 0$ and $F_1 = 1$. The well-known Binet formula (actually already known to de Moivre) expresses F_i as a linear combination of the zeroes $\phi \doteq 1.61803 > 0 > \hat{\phi}$ of the characteristic polynomial of the recurrence $X^2 - X - 1$:

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\phi - \hat{\phi}}.$$

Here the number $\phi = \frac{\sqrt{5}+1}{2}$ is popularly referred to as the *golden mean* or *golden ratio*.

Similarly, the "Tribonacci" numbers (the name is apparently due to Feinberg [3]; also see [9]) are defined by

$$T_i = T_{i-1} + T_{i-2} + T_{i-3}$$

with initial values $T_0 = T_1 = 0$ and $T_2 = 1$. Here we also have that T_i is a linear combination of $\eta_1^i, \eta_2^i, \eta_3^i$, where η_1, η_2, η_3 are the zeroes of the characteristic polynomial $X^3 - X^2 - X - 1$; see, e.g., [10]. Here

$$\eta_1 = \frac{1}{3}\left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right)$$

is the only real zero and $\eta_1 \doteq 1.839$.

The "Tetranacci" (aka "Tetrabonacci", "Quadranacci") numbers are defined analogously by

$$A_i = A_{i-1} + A_{i-2} + A_{i-3} + A_{i-4}$$

with initial values $A_0 = A_1 = A_2 = 0$ and $A_3 = 1$. Once again, the A_i can be expressed as a linear combination of the zeroes of the characteristic polynomial $X^4 - X^3 - X^2 - X - 1$; see, for example [6].

More generally, we can define the generalized Fibonacci sequence of order n by

$$G_i^{(n)} = G_{i-1}^{(n)} + \cdots + G_{i-n}^{(n)}$$

with appropriate initial terms. Here the associated characteristic polynomial is $X^n - X^{n-1} - \cdots - X - 1$. As is well-known [7, 8], this polynomial has a single positive zero α_n, which is strictly between 1 and 2. (The other zeroes are discussed in [12].) Table 1 gives decimal approximations of the first few dominant zeroes. Furthermore, as Dresden and Du have shown [1, Theorem 2], knowledge of α_n suffices to compute the ith generalized Fibonacci number of order n.

TABLE 1. Generalized Golden Means

n	α_n
2	1.61803398874989484820
3	1.83928675521416113255
4	1.92756197548292530426
5	1.96594823664548533719
6	1.98358284342432633039
7	1.99196419660503502110
8	1.99603117973541458982
9	1.99802947026228669866
10	1.99901863271010113866

It is natural to wonder how the generalized golden means α_n behave as $n \to \infty$. Dubeau [2] proved that $(\alpha_n)_{n\geq 2}$ is an increasing sequence that converges to 2. In fact, it is not hard to show, using the binomial theorem, that

$$2 - \frac{1}{2^n - \frac{n}{2} - \frac{n^2}{2^n}} < \alpha_n < 2 - \frac{1}{2^n - \frac{n}{2}}$$

for $n \geq 2$; see [5].

In this paper, we give three series that approximate α_n, $1/\alpha_n$, and $1/(2-\alpha_n)$ to any desired order. Remarkably, all three have similar forms.

Theorem 1.1. *Let $n \geq 2$, and define $\alpha = \alpha_n$, the positive real zero of $X^n - X^{n-1} - \cdots - X - 1$. Let $\beta = 1/\alpha$. Then*

(a)
$$\beta = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}}.$$

(b)
$$\alpha = 2 - 2\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k(n+1)}}.$$

(c)
$$\frac{1}{2-\alpha} = 2^n - \frac{n}{2} - \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)}{k+1}\frac{1}{2^{k(n+1)}}.$$

The proof is given in the next three sections. Our main tool is the classical Lagrange inversion formula; see, for example, [4, Section A.6, p. 732].

Theorem 1.2. *Let $\Phi(t)$ and $f(t)$ be formal power series, and suppose $t = z\Phi(t)$. If $\Phi(0) \neq 0$, we can write $t = t(z)$ as a formal power series in z. Then for integers $k \geq 1$ we have*

(a) $[z^k]t = \frac{1}{k}[t^{k-1}](\Phi(t))^k$;
(b) $[z^k]f(t) = \frac{1}{k}[t^{k-1}]f'(t)(\Phi(t))^k$;

where, as usual, $[z^k]t$ (resp., $[z^k]f(t)$) denotes the coefficient of z^k in the series for t (resp., $f(t)$).

2. A Series for β

In this section, we will prove Theorem 1.1 (a), namely:

$$\beta = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}}.$$

Proof. From

$$\alpha^n = \alpha^{n-1} + \cdots + \alpha + 1$$

we get

$$(1-\alpha)\alpha^n = 1 - \alpha^n$$

and hence,

$$\alpha^{n+1} - 2\alpha^n + 1 = 0. \tag{2.1}$$

Recalling that $\beta = 1/\alpha$ we get

$$\beta = \frac{1}{2} + \frac{1}{2}\beta^{n+1}. \tag{2.2}$$

Let $\Phi(t) = (t + \frac{1}{2})^{n+1}$ and

$$t = z\Phi(t), \tag{2.3}$$

as in the hypothesis of Theorem 1.2. We notice that $t = \beta - \frac{1}{2}$ and $z = \frac{1}{2}$ is a solution to equation (2.3) as shown in equation (2.2). From the Lagrange inversion formula and the binomial theorem, we get

$$[z^k]t = \frac{1}{k}[t^{k-1}]\Big(t + \frac{1}{2}\Big)^{k(n+1)} = \frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)+1-k}}.$$

So

$$t = \sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)+1-k}}z^k.$$

In particular, at $z = \frac{1}{2}$ and $t = \beta - \frac{1}{2}$, we get

$$\beta = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}},$$

as required. □

3. A Series for α

In this section, we will prove Theorem 1.1 (b), namely:

$$\alpha = 2 - 2\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k(n+1)}}.$$

This formula was previously discovered in 1998 by Wolfram [11, Theorem 3.9].

Proof. From (2.1) we get
$$\alpha^n(\alpha-2)+1=0$$
and so
$$2-\alpha=\alpha^{-n}. \tag{3.1}$$
Let $\Phi(t)=(1-\frac{t}{2})^{-n}$ and
$$t=z\Phi(t)$$
as in the hypothesis of Theorem 1.2. We observe that $t=2-\alpha$ and $z=2^{-n}$ is a solution, as shown in equation (3.1). Using the Lagrange inversion formula again, we find
$$[z^k]t=\frac{1}{k}[t^{k-1}]\Big(1-\frac{t}{2}\Big)^{-kn}=\frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k-1}}.$$
Therefore,
$$t=\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)-2}{k-1}z^k\frac{1}{2^{k-1}}.$$
In particular, evaluating this at $t=2-\alpha$ and $z=2^{-n}$ gives
$$2-\alpha=\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)-2}{k-1}2^{-nk}\frac{1}{2^{k-1}},$$
or
$$\alpha=2-2\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k(n+1)}},$$
giving us a series for α. □

4. A Series for $1/(2-\alpha)$

In this section we will prove Theorem 1.1 (c), namely:
$$\frac{1}{2-\alpha}=2^n-\frac{n}{2}-\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)}{k+1}\frac{1}{2^{k(n+1)}}.$$

Proof. Define
$$S(z)=-\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}z^k[t^{k+1}](1+t)^{k(n+1)}.$$
At $z=2^{-(n+1)}$, this gives
$$S\left(1/2^{n+1}\right)=-\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)}{k+1}\frac{1}{2^{k(n+1)}}.$$
Hence it suffices to show that
$$S\left(1/2^{n+1}\right)=-2^n+\frac{n}{2}+\frac{1}{2-\alpha}.$$
We see from equation (3.1) that
$$\frac{2}{\alpha}-1=\alpha^{-n-1}. \tag{4.1}$$
Let $t=z\Phi(t)$ as before. Further let
$$\Phi(t)=(1+t)^{n+1},\quad f'(t)=-\Phi^{-2}.$$

We see that $z = 1/2^{n+1}$ and $t = \frac{2}{\alpha} - 1$ is a solution to $t = z\Phi(t)$ by equation (4.1).

To get a series for $1/(2-\alpha)$, we start from the Lagrange inversion formula, part (b), to get

$$f(t) = f(0) + \sum_{k\geq 1} \frac{1}{k} z^k [t^{k-1}](\Phi(t))^k f'(t).$$

Differentiating with respect to z gives

$$\frac{d}{dz} f(t) = \frac{dt}{dz} \cdot f'(t) = \sum_{k\geq 1} z^{k-1}[t^{k-1}](\Phi(t))^k f'(t).$$

Using $z = t/\Phi(t)$ we get

$$\frac{dz}{dt} = \frac{\Phi(t) - t\Phi'(t)}{\Phi(t)^2}$$

and so

$$\frac{dt}{dz} = \frac{\Phi(t)^2}{\Phi(t) - t\Phi'(t)}.$$

This gives us

$$\begin{aligned}\frac{\Phi^2}{\Phi - t\Phi'} \cdot f'(t) &= \sum_{k\geq 1} z^{k-1}[t^{k-1}](\Phi(t))^k f'(t) \\ &= [t^0]\Phi(t) f'(t) + z^1[t^1](\Phi(t))^2 f'(t) + \sum_{k\geq 1} z^{k+1}[t^{k+1}](\Phi(t)^k)(\Phi(t))^2 f'(t).\end{aligned}$$

Using the fact that $f'(t) = -\frac{1}{\Phi^2}$ we get

$$-\frac{1}{\Phi - t\Phi'} = -1 - \sum_{k\geq 1} z^{k+1}[t^{k+1}](\Phi(t))^k.$$

Observing that $S'(z) = -\frac{1}{2}\sum_{k\geq 1} z^{k-1}[t^{k+1}](1+t)^{k(n+1)}$, this simplifies to

$$2z^2 S'(z) = 1 - \frac{1}{\Phi - t\Phi'}.$$

Thus,

$$S'(z) = \frac{1}{2z^2} - \frac{1}{\Phi - t\Phi'}\frac{\Phi^2}{2t^2},$$

so

$$S(z) = -\frac{1}{2z^1} - \int \frac{1}{\Phi - t\Phi'}\frac{\Phi^2}{2t^2} dz = -\frac{1}{2z} - \int \frac{\Phi - t\Phi'}{\Phi^2}\frac{1}{\Phi - t\Phi'}\frac{\Phi^2}{2t^2} dt$$

and

$$S(z) = -\frac{1}{2z} - \int dt \frac{1}{2t^2} = -\frac{1}{2z} + \frac{1}{2t} + C.$$

In order to compute the integration constant C, we note that $S(0) = 0$. Then

$$C = \frac{1}{2}\lim_{z\to 0}\left[\frac{1}{z} - \frac{1}{t}\right] = \frac{1}{2}\lim_{t\to 0}\frac{\Phi - 1}{t} = \frac{1}{2}\lim_{t\to 0}\frac{(1+t)^{n+1} - 1}{t} = \frac{n+1}{2}$$

and

$$S(z) = -\frac{1}{2z} + \frac{1}{2t} + \frac{n+1}{2}.$$

Evaluating at $z = 1/2^{n+1}$ and $t = \frac{2}{\alpha} - 1$ we have

$$S(1/2^{n+1}) = -\frac{1}{2z} + \frac{1}{2t} + \frac{n+1}{2} = -2^n + \frac{\alpha}{2(2-\alpha)} + \frac{n+1}{2} = -2^n + \frac{1}{2-\alpha} + \frac{n}{2},$$

as required. $\square$

5. Speed of Convergence

The speed of convergence of the series in Theorem 1.1 is determined by the individual terms in the sequence. For example, consider the series for $1/\alpha$:

$$\beta = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}}.$$

The convergence depends upon the speed of convergence of

$$f_1(k,n)/2^{k(n+1)} := \frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}}.$$

Similarly, define

$$f_2(k,n)/2^{k(n+1)} := \frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k(n+1)}}.$$

$$f_3(k,n)/2^{k(n+1)} := \frac{1}{k}\binom{k(n+1)}{k+1}\frac{1}{2^{k(n+1)}}$$

based on the expansion of α and $1/(2-\alpha)$.

Notice that by Stirling's approximation we have

$$\begin{aligned}\lim_{k\to\infty}\log_2(f_1(k,n))/k &\approx \lim_{k\to\infty}\log_2(f_2(k,n))/k\\ &\approx \lim_{k\to\infty}\log_2(f_3(k,n))/k\\ &\approx (n+1)\log_2(n+1) - n\log_2(n),\end{aligned}$$

which, as $n \to \infty$, tends to $\log_2(n+1) + \frac{1}{\ln(2)}$.

Thus, for example, when $n = 2$ (corresponding to the Fibonacci case), we have

$$\log_2 f_i(k,n) \sim (3\log_2(3) - 2\log_2(2))k \sim (2.75489\cdots)k.$$

Since each term of the summation is of the form $f_i(k,n)/2^{k(n+1)}$, in the case $n = 2$, the kth term is approximately $2^{-.24511k}$. Thus, for example, 1000 terms of the series are expected to give at least 73 correct digits; in fact, it gives 77 or 78 depending on the series. Here by digits of accuracy, we mean $\lfloor -\log_{10}|\text{actual} - \text{estimate}|\rfloor$, which is the number of correct decimal digits after the decimal point. See Table 2 for a summation of various predictions versus actual accuracy.

We notice that convergence is much much faster for larger n.

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TABLE 2. Predicted and Actual Accuracy of Truncated Series

n	k	Predicted accuracy	Actual accuracy (α)	Actual accuracy ($1/\alpha$)	Actual accuracy ($1/(2-\alpha)$)
2	100	7	10	10	9
2	1000	73	78	78	77
2	10000	737	744	743	743
10	10	18	23	23	21
10	100	185	192	191	190
10	1000	1856	1864	1863	1862
100	2	55	87	86	83
100	10	279	311	311	307
100	100	2796	2830	2829	2826

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