STIRLING WITHOUT WALLIS

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Abstract. It is fairly easy to show that

$$n! \sim Cn^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty,$$

and it is then standard procedure to use Wallis' product to show that

$$C = \sqrt{2\pi}.$$

The purpose of this note is to show that there is an alternative route to determining C.

1. Introduction

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The purpose of this note is to show that there is an alternative route to determining C, and consequently a nonstandard way to derive Wallis' product.

2. The Usual Procedure, From Wallis to Stirling

If we let

$$u_n = n! \Big/ n^{n+\frac{1}{2}}e^{-n},$$

then

$$\frac{u_n}{u_{n-1}} \approx \exp\left\{-\frac{1}{12n^2}\right\},$$

from which it follows that

$$u_n \to C \text{ as } n \to \infty,$$

where C is a nonzero constant, and so

$$n! \sim Cn^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty.$$

Now, Wallis' product, which follows from the fact that

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

is a decreasing function of n, together with the facts that

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ \cdots \ \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \text{ if } n \text{ is odd,}$$
$$= \frac{n-1}{n} \cdot \frac{n-3}{n-1} \cdot \ \cdots \ \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ if } n \text{ is even}$$

says that

$$\frac{\pi}{4} = \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots,$$

or, equivalently,

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\cdots,$$

or, yet again,

$$\pi = \lim_{n\to\infty} \frac{2^{4n} n!^4}{n(2n)!^2}.$$

Taken together with

$$n! \ \sim \ Cn^{n+\frac{1}{2}} e^{-n},$$

this gives

$$\frac{C^2}{2} = \pi,$$

so

$$C = \sqrt{2\pi}$$

and

$$n! \ \sim \ \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

which is Stirling's formula.

For all this, see for example, [1, Vol. II, pp. 616–618].

3. Stirling Without Wallis

In this section we show that

$$n! \ \sim \ \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

without using Wallis' product.

We start with the series for e^n,

$$e^n = 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{n^k}{k!}.$$

The (equally) largest term in this expansion occurs when $k = n$, and is $H = \dfrac{n^n}{n!}$. Nearby terms are given by

$$\begin{aligned}\frac{n^{n+k}}{(n+k)!} &= H \cdot \frac{n^{n+k}}{(n+k)!} \Big/ \frac{n^n}{n!} \\ &= H \cdot \frac{n}{n+1} \cdots \frac{n}{n+k} \\ &= H \cdot \exp\left\{-\log\left(1+\frac{1}{n}\right) - \cdots - \log\left(1+\frac{k}{n}\right)\right\} \\ &\approx H \cdot \exp\left\{-\frac{1}{n} - \cdots - \frac{k}{n}\right\} \\ &\approx H \cdot \exp\left\{-\frac{k^2+k}{2n}\right\},\end{aligned}$$

for terms to the right of n, or by

$$\begin{aligned}\frac{n^{n-k}}{(n-k)!} &= H \cdot \frac{n^{n-k}}{(n-k)!} \Big/ \frac{n^n}{n!} \\ &= H \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \\ &= H \cdot \exp\left\{\log\left(1-\frac{1}{n}\right) + \cdots + \log\left(1-\frac{k-1}{n}\right)\right\} \\ &\approx H \cdot \exp\left\{-\frac{1}{n} - \cdots - \frac{k-1}{n}\right\} \\ &\approx H \cdot \exp\left\{-\frac{k^2-k}{2n}\right\}\end{aligned}$$

for terms to the left of n.

So the distribution function is close to

$$\begin{aligned}f(x) &= H \exp\left\{-\frac{(x-n)^2+(x-n)}{2n}\right\} \\ &= H \exp\left\{-\frac{(x-n+\frac{1}{2})^2-\frac{1}{4}}{2n}\right\} \\ &= H \exp\left\{-\frac{(x-(n-\frac{1}{2}))^2}{2n} + \frac{1}{8n}\right\}.\end{aligned}$$

Thus the terms are distributed roughly normally about the mean $(n-\frac{1}{2})$ with standard deviation σ given by

$$\sigma^2 = n,$$

or

$$\sigma = \sqrt{n}.$$

It follows that

$$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} \approx H e^{\frac{1}{8n}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2n}\right\} \, dx \approx H\sigma\sqrt{2\pi} = \sqrt{2\pi n}\,\frac{n^n}{n!},$$

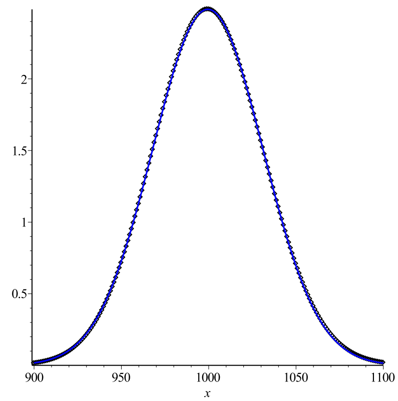

FIGURE 1. The case $n = 1000$, showing the points $(k, \frac{n^k}{k!})$ for $900 \le k \le 1100$, together with the normal $y = \frac{n^n}{n!} \exp\left\{-\frac{\left(x-(n-\frac{1}{2})\right)^2}{2n} + \frac{1}{8n}\right\}$.

and so

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Of course, this argument can be tightened (with a fair bit of trouble) to give

$$n! \ \sim \ \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

From this we easily find that

$$\lim_{n\to\infty} \frac{2^{4n} n!^4}{n(2n)!^2} = \pi,$$

which is Wallis' product.

References

[1] T. M. Apostol, *Calculus*, Xerox College Publishing, Waltham, Massachusetts, 1969.

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