CONDITIONS GOVERNING CROSS-FAMILY MEMBER EQUALITY IN A PARTICULAR CLASS OF POLYNOMIAL FAMILIES

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Abstract. There exists a class of polynomial families associated with integer sequences whose ordinary generating functions have quadratic governing equations with functional (polynomial) coefficients. In this paper we give necessary and sufficient conditions under which corresponding members of two such polynomial families are equal, with some supporting examples provided.

1. Introduction

Let $A(x), B(x), C(x) \in \mathbb{Z}[x]$, and suppose the (ordinary) generating function $T(x)$ of a sequence of integers satisfies a general quadratic governing equation

$$0 = A(x)T^2(x) + B(x)T(x) + C(x). \tag{1.1}$$

The functional coefficients $A(x), B(x), C(x)$ can be considered to give rise to a family of associated polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, defined as

$$\begin{aligned}\alpha_n(x) &= \alpha_n(A(x), B(x), C(x))\\ &= (1,0)\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \geq 0,\end{aligned} \tag{1.2}$$

and for which the first few polynomials are, explicitly,

$$\begin{aligned}\alpha_0(x) &= 1,\\ \alpha_1(x) &= -B(x),\\ \alpha_2(x) &= B^2(x) - A(x)C(x),\\ \alpha_3(x) &= 2A(x)B(x)C(x) - B^3(x),\\ \alpha_4(x) &= B^4(x) - 3A(x)B^2(x)C(x) + A^2(x)C^2(x),\\ \alpha_5(x) &= 4A(x)B^3(x)C(x) - 3A^2(x)B(x)C^2(x) - B^5(x),\end{aligned} \tag{1.3}$$

etc. A general closed form is [1, Eq. (52), p. 24]

$$\alpha_n(x) = \frac{1}{2^{n+1}} \frac{[-B(x) + \rho(x)]^{n+1} - [-B(x) - \rho(x)]^{n+1}}{\rho(x)}, \tag{1.4}$$

where $\rho(x) = \rho(A(x), B(x), C(x)) = \sqrt{B^2(x) - 4A(x)C(x)}$ is the 'discriminant' function for (1.1). Some simplified instances of the general polynomial $\alpha_n(x)$ are

$$\begin{aligned}\alpha_n(A(x), B(x), 0) &= [-B(x)]^n = \alpha_n(0, B(x), C(x)),\\ \alpha_n(A(x), 0, C(x)) &= \begin{cases} 0 & n \text{ (odd)} \geq 1 \\ [-A(x)C(x)]^{n/2} & n \text{ (even)} \geq 0, \end{cases}\end{aligned} \tag{1.5}$$

whose proofs are omitted (simple reader exercise). From hereon we will assume, for convenience, that the arguments $A(x), B(x), C(x)$ of $\alpha_n(x)$ are non-zero.

In [4] a new non-linear identity was established for this class of families on which, as seen from the references therein, a considerable amount of work has been conducted and results given for the particular Catalan, (Large) Schröder and Motzkin polynomial families (described by respective instances $P_n(x) = \alpha_n(x, -1, 1)$, $S_n(x) = \alpha_n(x, x-1, 1)$ and $M_n(x) = \alpha_n(x^2, x-1, 1)$) that form a natural grouping and are associated with their namesake sequences, see OEIS Sequence Nos. A000108, A006318 and A001006 [5]. Specializations of the general polynomial $\alpha_n(A(x), B(x), C(x))$ in the context of the Fibonacci sequence, as noted by the referee of [4], provide an interesting observation and in turn motivation for further analysis here in which we explore conditions under which corresponding members of two polynomial families are equal. Accordingly, results are formulated and examples included.

2. A Conjecture and Formal Results

2.1. **A Conjecture.** Let $\{F_n\}_0^\infty = \{1, 1, 2, 3, 5, \ldots\} = \{F_0, F_1, F_2, F_3, F_4, \ldots\}$ denote the Fibonacci sequence, with $(n+1)$th term F_n. We already know [3, Eq. (13), p. 47] that, setting $A(x) = -1$, $B(x) = C(x) = 1$, the polynomial $\alpha_n(-1, 1, 1)$ evaluates to $(-1)^n F_n$ $(n \geq 0)$, a result easily checked by hand for $n = 0, \ldots, 5$ using (1.3). With a sign change in the polynomial arguments it is immediate via (1.2) that $\alpha_n(1, -1, -1) = F_n$ $(n \geq 0)$ (of course knowing the closed form $[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}]/2^{n+1}\sqrt{5}$ for F_n, both results follow from (1.4)), and we find, too, that

$$\alpha_n(1, -1, -1) = \alpha_n(-1, -1, 1) = F_n, \quad n \geq 0. \tag{2.1}$$

This is an interesting result since it translates as the matrix identity

$$(1,0)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1,0)\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \geq 0, \tag{2.2}$$

which is not obvious. Equation (2.2), together with other similar examples such as

$$\begin{aligned}
(1,0)\begin{pmatrix} -2 & 4 \\ -5 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (1,0)\begin{pmatrix} -2 & -2 \\ 10 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
(1,0)\begin{pmatrix} 1 & -3 \\ -4 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (1,0)\begin{pmatrix} 1 & 2 \\ 6 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1,0)\begin{pmatrix} 1 & 12 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
(1,0)\begin{pmatrix} -5 & -4 \\ 4 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (1,0)\begin{pmatrix} -5 & \sqrt{2} \\ -8\sqrt{2} & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned} \tag{2.3}$$

(all checked computationally for many values of $n \geq 0$) lead naturally to a conjecture, a quick glance at the structure of the polynomials in (1.3) suggesting that this is not an unreasonable one to make.

Let $A(x), A'(x), B(x), B'(x), C(x), C'(x) \in \mathbb{Z}[x]$. Noting that $\alpha_0 = 1$ for $A(x), B(x), C(x)$ arbitrary, and that $\alpha_1 = \alpha_1(B(x))$, the *conjecture is that, for* $n \geq 2$, $\alpha_n(A(x), B(x), C(x)) = \alpha_n(A'(x), B'(x), C'(x)) \Leftrightarrow B(x) = B'(x)$ *and* $A(x)C(x) = A'(x)C'(x)$.

As far as producing an argument to confirm the proposed condition of necessity, this is straightforward since we have a prior result on which to draw. We assume that $\alpha_n(A(x), B(x), C(x)) = \alpha_n(A'(x), B'(x), C'(x))$ for some $n \geq 2$. Applying directly the somewhat unusual

Theorem 3 of [1, p. 21], which states that

$$\alpha_n(A(x),B(x),C(x)) = [-B(x)]^n P_n\left(\frac{A(x)C(x)}{B^2(x)}\right), \quad n \geq 0, \tag{2.4}$$

in terms of the $(n+1)$th Catalan polynomial $P_n(x)$, we infer $B^n(x)P_n(A(x)C(x)/B^2(x)) = B'^n(x)P_n(A'(x)C'(x)/B'^2(x))$; this holds for $B(x) = B'(x)$, $A(x)C(x) = A'(x)C'(x)$. Note that (2.4) follows readily from the interesting result [2, Eq. (21), p. 142]

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n = \begin{pmatrix} P_n(-xy) & xP_{n-1}(-xy) \\ yP_{n-1}(-xy) & xyP_{n-2}(-xy) \end{pmatrix} \tag{2.5}$$

giving the evaluation of the nth power of the matrix $\left(\begin{smallmatrix} 1 & x \\ y & 0 \end{smallmatrix}\right)$ as a matrix whose entries involve Catalan polynomials having common argument $-xy$ throughout. Employing (2.5) under the assumption that $B(x) = B'(x)$, $A(x)C(x) = A'(x)C'(x)$, it is easy enough (and so omitted) to conclude that $\alpha_n(A(x),B(x),C(x)) = \alpha_n(A'(x),B'(x),C'(x))$ for $n \geq 2$ in consequence from a simple reverse argument, which is the suggested condition of sufficiency.

2.2. **Formal Results.** While the sufficiency condition put forward is indeed correct, the necessary condition as stated is false for it is oversimplistic through the sole use of (2.4). We formally establish, therefore, conditions under which corresponding members of different polynomial families are equal as separate statements in terms of implications, and we do so in a more rigorous, and hence more satisfactory, fashion. Note that the actual definition of the general Catalan polynomial plays a role in the necessary condition.

Theorem 2.1. *(Sufficiency.) If $B(x) = B'(x)$ and $A(x)C(x) = A'(x)C'(x)$ then, for $n \geq 2$, $\alpha_n(A(x),B(x),C(x)) = \alpha_n(A'(x),B'(x),C'(x))$.*

Proof. We prove this by showing, equivalently, that $\alpha_n(A(x),B(x),C(x)) = \alpha_n(-A(x)C(x)/z, B(x), -z)$ $(n \geq 2)$ for arbitrary $z = z(x) \neq 0$, which we achieve through a constructive first principles approach (note that we have an alternative proof, given in the Appendix).

Suppose $n \geq 2$, and let $\mathbf{M}(x)$ be the matrix

$$\mathbf{M}(x) = \mathbf{M}(A(x),B(x),C(x)) = \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix} \tag{T.1}$$

integral to the definition (1.2) of $\alpha_n(x) = (1,0)\,\mathbf{M}^n(x)\binom{1}{0}$. Defining two other matrices

$$\mathbf{L}(x;z) = \begin{pmatrix} -B(x) & -A(x)C(x)/z \\ z & 0 \end{pmatrix}, \qquad \mathbf{K}(x;z) = \begin{pmatrix} B(x)/C(x) & A(x)/z \\ 1 & 0 \end{pmatrix}, \tag{T.2}$$

we begin by observing the judicious decomposition

$$\mathbf{M}(x) = \mathbf{K}(x;z)\mathbf{L}(x;z)\mathbf{K}^{-1}(x;z), \tag{T.3}$$

from which

$$\begin{aligned} \alpha_n(x) &= (1,0)\mathbf{M}^n(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1,0)[\mathbf{K}(x;z)\mathbf{L}(x;z)\mathbf{K}^{-1}(x;z)]^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1,0)\mathbf{K}(x;z)\cdot\mathbf{L}^n(x;z)\cdot\mathbf{K}^{-1}(x;z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{T.4}$$

We suppose $\mathbf{L}^n(x;z)$ has form

$$\mathbf{L}^n(x;z) = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \tag{T.5}$$

for assumed $\alpha^* = \alpha^*(x;z), \ldots, \delta^* = \delta^*(x;z)$. Then, with

$$(1,0)\mathbf{K}(x;z) = (B(x)/C(x), A(x)/z), \quad \mathbf{K}^{-1}(x;z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ z/A(x) \end{pmatrix}, \tag{T.6}$$

(T.4) delivers

$$\alpha_n(A(x), B(x), C(x)) = \alpha_n(x; z(x)) = \frac{B(x)\beta^* z}{A(x)C(x)} + \delta^*. \tag{T.7}$$

We now express the self-satisfying equation $\mathbf{L}^{n+1}(x;z) = \mathbf{L}^{n+1}(x;z)$ as

$$\mathbf{L}(x;z) \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \mathbf{L}(x;z), \tag{T.8}$$

or

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \mathbf{L}(x;z) \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} - \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \mathbf{L}(x;z) \\ &= \begin{pmatrix} Q_1(x;z) & Q_2(x;z) \\ Q_3(x;z) & Q_4(x;z) \end{pmatrix}, \end{aligned} \tag{T.9}$$

where

$$\begin{aligned} Q_1(x;z) &= -(A(x)C(x)\gamma^*/z + \beta^* z), \\ Q_2(x;z) &= A(x)C(x)(\alpha^* - \delta^*)/z - B(x)\beta^*, \\ Q_3(x;z) &= (\alpha^* - \delta^*)z + B(x)\gamma^*, \\ Q_4(x;z) &= A(x)C(x)\gamma^*/z + \beta^* z, \end{aligned} \tag{T.10}$$

after some elementary algebra. Equating entries across (T.9), we have first that $0 = Q_1(x;z) = -Q_4(x;z) \Rightarrow \gamma^* = -\beta^* z^2/(A(x)C(x))$, so that $0 = Q_3(x;z) \Rightarrow$

$$\alpha^* = \delta^* - \frac{B(x)\gamma^*}{z} = \delta^* - \frac{B(x)}{z}\left(-\frac{\beta^* z^2}{A(x)C(x)}\right) = \delta^* + \frac{B(x)\beta^* z}{A(x)C(x)}. \tag{T.11}$$

Thus,[1] from (and in order) (1.2),(T.2),(T.5),(T.11) and (T.7), we can make the deduction

$$\begin{aligned}
\alpha_n(-A(x)C(x)/z, B(x), -z) &= (1,0)\begin{pmatrix} -B(x) & -A(x)C(x)/z \\ z & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (1,0)\mathbf{L}^n(x;z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (1,0)\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \alpha^* \\
&= \delta^* + \frac{B(x)\beta^* z}{A(x)C(x)} \\
&= \alpha_n(A(x), B(x), C(x)), \qquad \text{(T.12)}
\end{aligned}$$

as required. $\square$

A simple counter-example to the conjectured condition of necessity demonstrates a need for its revision for we find, with $B(x) = B'(x) = (x+1)(x+2)$, $A(x) = A'(x) = x+1$, $C(x) = x+2$, $C'(x) = 3x^3 + 15x^2 + 23x + 10$ (for which clearly $A(x)C(x) \neq A'(x)C'(x)$), that $\alpha_4(A(x), B(x), C(x)) = (x+1)^2(x+2)^2(x^4 + 6x^3 + 10x^2 + 3x - 1) = \alpha_4(A'(x), B'(x), C'(x))$.

Theorem 2.2. *(Necessity.) Let $n \geq 2$ and define a function*

$$D_n(A(x), A'(x), B(x), C(x), C'(x)) = \sum_{i=1}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} \left(\frac{-1}{B^2(x)}\right)^i \sum_{j=0}^{i-1} [A(x)C(x)]^j [A'(x)C'(x)]^{i-1-j}.$$

If two corresponding polynomial family members $\alpha_n(A(x), B(x), C(x))$ and $\alpha_n(A'(x), B(x), C'(x))$ are equal, then either $A(x)C(x) = A'(x)C'(x)$ or $D_n(A(x), A'(x), B(x), C(x), C'(x)) = 0$.

Proof. Consider, for $n \geq 2$, the equality of two polynomials $\alpha_n(A(x), B(x), C(x))$ and $\alpha_n(A'(x), B(x), C'(x))$, which necessarily means $P_n(A(x)C(x)/B^2(x)) = P_n(A'(x)C'(x)/B^2(x))$ by (2.4). Appealing to the form $P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-x)^i$ of the general Catalan polynomial (see this and other formats in [1, Eqs. (9)-(11), p. 6], together with $P_0(x), \ldots, P_7(x)$ listed [1, Eq. (13), p. 7][2]), this implied condition now becomes

$$\begin{aligned}
0 &= \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i}(-1)^i \left[\left(\frac{A(x)C(x)}{B^2(x)}\right)^i - \left(\frac{A'(x)C'(x)}{B^2(x)}\right)^i\right] \\
&= \sum_{i=1}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i}(-1)^i \left[\left(\frac{A(x)C(x)}{B^2(x)}\right)^i - \left(\frac{A'(x)C'(x)}{B^2(x)}\right)^i\right]. \qquad \text{(T.13)}
\end{aligned}$$

[1] Although not actually used in the proof, for the sake of completeness we note here that $Q_2(x;z) = A(x)C(x)(\alpha^* - \delta^*)/z - B(x)\beta^* = A(x)C(x)[-B(x)\gamma^*/z]/z - B(x)\beta^* = -A(x)B(x)C(x)\gamma^*/z^2 - B(x)\beta^* = -A(x)B(x)C(x)[-\beta^* z^2/(A(x)C(x))]/z^2 - B(x)\beta^* = B(x)\beta^* - B(x)\beta^* = 0$, and hence see that the system of equations given by (T.9) and (T.10) is a consistent one.

[2] We note in passing, for the interested reader, that [1, Eqs. (29),(31), p. 17] give the same first few (Large) Schröder and Motzkin polynomials in explicit form.

Using the well-known result $s^i - t^i = (s-t)\sum_{j=0}^{i-1} s^j t^{i-1-j}$ with $s = A(x)C(x)/B^2(x)$, $t = A'(x)C'(x)/B^2(x)$, this reads

$$\begin{aligned} 0 &= \sum_{i=1}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i}(-1)^i \cdot \left(\frac{A(x)C(x) - A'(x)C'(x)}{B^{2i}(x)}\right) \sum_{j=0}^{i-1} [A(x)C(x)]^j [A'(x)C'(x)]^{i-1-j} \\ &= [A(x)C(x) - A'(x)C'(x)] D_n(A(x), A'(x), B(x), C(x), C'(x)); \end{aligned} \tag{T.14}$$

thus, for (T.14) to be satisfied then either $D_n(A(x), A'(x), B(x), C(x), C'(x)) = 0$ or $A(x)C(x) = A'(x)C'(x)$. □

We finish with some examples and remarks.

Examples (Theorem 2.2). By way of illustration we see that in the earlier example for which $A(x) = A'(x) = x + 1$, $B(x) = (x+1)(x+2)$, $C(x) = x + 2$, $C'(x) = 3x^3 + 15x^2 + 23x + 10$, then we observe the specific equality $\alpha_4(A(x), B(x), C(x)) = \alpha_4(A'(x), B(x), C'(x))$ due to the fact that $D_4(A(x), A'(x), B(x), C(x), C'(x)) = \sum_{i=1}^{2} \binom{4-i}{i} [-1/B^2(x)]^i \sum_{j=0}^{i-1} [A(x)C(x)]^j [A'(x)C'(x)]^{i-1-j} = [A(x)C(x) + A'(x)C'(x) - 3B^2(x)]/B^4(x)$ vanishes in this instance. A similar example is described by the equality of polynomials $\alpha_4(5, x, x^2)$ and $\alpha_4(-2x, x, x)$ from different families (both being $11x^4$) since $D_4(5, -2x, x, x^2, x) = [(5)(x^2) + (-2x)(x) - (3)(x^2)]/x^4 = 0$.

We also see that, for example,

$$\begin{aligned} \alpha_5(2(x^2-1), -\sqrt{15/2}x^2, 5(x^2+1)) &= \alpha_5(10, -\sqrt{15/2}x^2, 1) \\ &= 75\sqrt{30}x^2(2 - 2x^4 + 3x^8/8), \end{aligned} \tag{2.6}$$

since $D_5(2(x^2-1), 10, -\sqrt{15/2}x^2, 5(x^2+1), 1) = 0$ (where, in general, $D_5(A(x), A'(x), B(x), C(x), C'(x)) = [3A(x)C(x) + 3A'(x)C'(x) - 4B^2(x)]/B^4(x)$).

Remark 2.1. We observe that $\alpha_{2,3}(A(x), B(x), C(x)) = \alpha_{2,3}(A'(x), B(x), C'(x))$ only when $A(x)C(x) = A'(x)C'(x)$, for neither $D_2(A(x), A'(x), B(x), C(x), C'(x)) = -1/B^2(x)$, nor $D_3(A(x), A'(x), B(x), C(x), C'(x)) = -2/B^2(x)$, are ever zero; this observation is immediate also from the definition of $\alpha_2(x)$ and $\alpha_3(x)$ in (1.3).

Finally, further examples confirming the correctness of Theorems 2.1 and 2.2 are given, which is felt to be instructive. Choose $A(x) = 2x$, $A'(x) = 8x^2$, $C(x) = 32x^3$, $C'(x) = 8x^2$. With (a) $B(x) = B_a(x) = 8\sqrt{2/3}x^2$ then $D_4(2x, 8x^2, B_a(x), 32x^3, 8x^2) = 0$, while with (b) $B(x) = B_b(x) = 4\sqrt{6}x^2$ then $D_5(2x, 8x^2, B_b(x), 32x^3, 8x^2) = 0$, but $\alpha_n(2x, B_{a,b}(x), 32x^3) = \alpha_n(8x^2, B_{a,b}(x), 8x^2)$ in both (a),(b) cases for *every* $n \geq 2$ because the sufficient, and dominating, condition $A(x)C(x) = A'(x)C'(x)$ for this to occur is also satisfied.

Remark 2.2. For completeness we note that in this final example, and that of (2.6), we have drawn $B(x)$ from $\mathbb{R}[x]$. This indicates that the theorems in fact hold in a wider context than stated in terms of the sets from which $A(x), B(x), C(x)$ in (1.1) may belong.

Remark 2.3. It is worth pointing out that no closed form exists for the double sum D_n. Rewriting the inner sum (over j) as $[(AC)^i - (A'C')^i]/(AC - A'C')$, and denoting the summand of D_n as the function $s_i(n, A, A', B, C, C')$, it is found that the ratio s_{i+1}/s_i is not a rational function of i (in the sense that the summand is not a so called hypergeometric term), and thus D_n cannot be summed to a closed form.

3. Summary

In this paper we have developed further analysis on a particular class of polynomial families associated with integer sequences whose ordinary generating functions have quadratic governing equations with polynomial coefficients; in particular, necessary and sufficient conditions under which corresponding members of different polynomial families are equal have been formulated, and results checked with some illustrative examples.

Future study will hopefully reveal more mathematical properties of these polynomials, associated with which a number of open problems remain.

Appendix

Here we reprove the equivalent result for Theorem 2.1.

Proof. Define

$$\mathbf{P}(x;z) = \begin{pmatrix} -z/C(x) & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.1}$$

with $\mathbf{L}(x;z)$ as in (T.2). Then we see

$$\mathbf{M}(x) = \mathbf{P}(x;z)\mathbf{L}(x;z)\mathbf{P}^{-1}(x;z), \tag{A.2}$$

and so

$$\begin{aligned}
\alpha_n(A(x), B(x), C(x)) &= (1,0)\mathbf{M}^n(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (1,0)\mathbf{P}(x;z)\cdot\mathbf{L}^n(x;z)\cdot\mathbf{P}^{-1}(x;z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (-z/C(x),0)\cdot\begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}\cdot\begin{pmatrix} -C(x)/z \\ 0 \end{pmatrix} \\
&= \hat{\alpha},
\end{aligned} \tag{A.3}$$

assuming that $\mathbf{L}^n(x;z)$ has the form

$$\mathbf{L}^n(x;z) = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix} \tag{A.4}$$

with $\hat{\alpha} = \hat{\alpha}(x;z), \ldots, \hat{\delta} = \hat{\delta}(x;z)$. Making the simple observation that

$$(1,0)\begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-z/C(x),0)\begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}\begin{pmatrix} [-z/C(x)]^{-1} \\ 0 \end{pmatrix}, \tag{A.5}$$

however, we can immediately infer that $\hat{\alpha} = \alpha_n(-A(x)C(x)/z, B(x), -z)$ and the proof is complete via (A.3) as required. $\square$

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