A NEW BINARY BBP-TYPE FORMULA FOR $\sqrt{5}\log\phi$

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Abstract. Previously only a base 5 BBP-type formula is known for $\sqrt{5}\log\phi$, where $\phi = (\sqrt{5}+1)/2$, the golden ratio, (i.e. Formula 83 of the April 2013 edition of Bailey's Compendium of BBP-type formulas). In this paper we derive a new binary BBP-type formula for this constant. The formula is obtained as a particular case of a BBP-type formula for a family of logarithms.

1. Introduction

A Bailey-Borwein-Ploufe (BBP) type formula for a mathematical constant c has the form

$$c = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s}\,,$$

where s, b, and l are integers: the degree, base and length of the formula, respectively and the a_j are rational numbers.

Such a formula allows the extraction of the individual base b digits of a mathematical constant without the need to compute the previous digits. The original BBP formula, discovered in 1996 [4], allows the extraction of the binary or hexadecimal digits of the constant π. Many such formulas have since been discovered and can be found in Bailey's Online Compendium of BBP-type formulas [3] and in the references therein. Another online Compendium is also being maintained by the CARMA Institute [2]. At the time of writing this paper only a base 5 BBP-type formula is known for the mathematical constant $\sqrt{5}\log\phi$, where $\phi = (\sqrt{5}+1)/2$ is the golden ratio. This formula is listed as Formula 83 in the current edition of Bailey's Compendium.

In this present paper we derive a new binary BBP-type formula for $\sqrt{5}\log\phi$. The formula is presented as a particular case of a more general BBP-type formula derived for a family of logarithms.

2. Notation

The first degree polylogarithm function, which we employ in this paper, is defined by

$$\mathrm{Li}_1[z] = \sum_{r=1}^{\infty} \frac{z^r}{r} = -\log(1-z), \quad |z| \le 1,\ z \ne 1\,.$$

For $q, x \in \mathbb{R}$, we have the identities

$$\arctan\left(\frac{q\sin x}{1-q\cos x}\right) = \mathrm{Im}\ \mathrm{Li}_1\left[q\exp(ix)\right] = \sum_{r=1}^{\infty} \frac{q^r \sin rx}{r} \tag{2.1}$$

and

$$-\frac{1}{2}\log\left(1-2q\cos x+q^2\right)=\operatorname{Re}\operatorname{Li}_1\left[q\exp(ix)\right]=\sum_{k=1}^{\infty}\frac{q^r\cos rx}{r}. \tag{2.2}$$

The BBP-type formulas in this paper will be given in the standard notation for BBP-type formulas, introduced in [5]:

$$\sum_{k=0}^{\infty}\frac{1}{b^k}\sum_{j=1}^{l}\frac{a_j}{(kl+j)^s}\equiv P(s,b,l,A),$$

where s, b, and l are integers, and $A=(a_1,a_2,\ldots,a_l)$ is a vector of integers.

3. A General BBP-Type Formula for a Family of Logarithms

Theorem. *For any nonzero integer t, the following BBP-type formula holds*

$$\begin{aligned}
&\sqrt{5}\,\tanh^{-1}\left\{\left(\frac{1-t+2t^2}{1-t+3t^2-2t^3+4t^4}\right)t\sqrt{5}\right\}\\
&=\frac{5}{2^{20}t^{39}}P(1,2^{20}t^{40},40,(2^{19}\,t^{38},0,2^{18}\,t^{36},2^{18}\,t^{35},0,0,-2^{16}\,t^{32},2^{16}\,t^{31},2^{15}\,t^{30},\\
&\qquad 0,-2^{14}\,t^{28},-2^{14}\,t^{27},2^{13}\,t^{26},0,0,-2^{12}\,t^{23},-2^{11}\,t^{22},0,-2^{10}\,t^{20},0,-2^{9}\,t^{18},\\
&\qquad 0,-2^{8}\,t^{16},-2^{8}\,t^{15},0,0,2^{6}\,t^{12},-2^{6}\,t^{11},-2^{5}\,t^{10},0,2^{4}\,t^{8},2^{4}\,t^{7},-2^{3}\,t^{6},0,0,\\
&\qquad\quad 2^{2}\,t^{3},2\,t^{2},0,1,0)).
\end{aligned} \tag{3.1}$$

Proof. Consider the following identity which holds for $t\in\mathbb{R}$, $t\neq 0$:

$$\begin{aligned}
&\tanh^{-1}\left\{\left(\frac{1-t+2t^2}{1-t+3t^2-2t^3+4t^4}\right)t\sqrt{5}\right\}\\
&=\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{t\sqrt{2}}\exp\left(\frac{i\pi}{20}\right)\right]-\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{t\sqrt{2}}\exp\left(\frac{7i\pi}{20}\right)\right]\\
&\quad+\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{t\sqrt{2}}\exp\left(\frac{9i\pi}{20}\right)\right]-\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{t\sqrt{2}}\exp\left(\frac{17i\pi}{20}\right)\right].
\end{aligned} \tag{3.2}$$

It is straightforward to verify equation (3.2) by the use of the first equality in equation (2.2). Using the second equality in equation (2.2) and trigonometric addition rules, equation (3.2) can also be written as

$$\begin{aligned}
\tanh^{-1}\left\{\left(\frac{1-t+2t^2}{1-t+3t^2-2t^3+4t^4}\right)t\sqrt{5}\right\}&=\sum_{r=1}^{\infty}\frac{4}{r\sqrt{2^r t^{2r}}}\sin\left(\frac{r\pi}{5}\right)\sin\left(\frac{2r\pi}{5}\right)\cos\left(\frac{r\pi}{4}\right)\\
&=\sum_{r=1}^{\infty}\frac{1}{r\sqrt{2^r t^{2r}}}f(r),
\end{aligned} \tag{3.3}$$

where we have defined a periodic function

$$f(r)=4\sin\left(\frac{r\pi}{5}\right)\sin\left(\frac{2r\pi}{5}\right)\cos\left(\frac{r\pi}{4}\right),\ r\in\mathbb{Z}. \tag{3.4}$$

Since $f(r)\in\left\{0,\pm\sqrt{5}/\sqrt{2},\pm\sqrt{5}\right\}$ and since $f(40k+j)=f(j)$ for integers k and j, we can convert the above single sum to an equivalent double sum by setting $r=40k+j$ in equation (3.3), obtaining

$$\begin{aligned}
\tanh^{-1}\left\{\left(\frac{1-t+2t^2}{1-t+3t^2-2t^3+4t^4}\right)t\sqrt{5}\right\} &= \sum_{k=0}^{\infty}\left\{\frac{1}{(2^{20}t^{40})^k}\sum_{j=1}^{40}\left(\frac{f(j)}{t^j}\frac{1}{\sqrt{2^j}}\frac{1}{40k+j}\right)\right\} \\
&= \frac{\sqrt{5}}{2^{20}t^{39}}\sum_{k=0}^{\infty}\left\{\frac{1}{(2^{20}t^{40})^k}\sum_{j=1}^{40}\left(\frac{a_j}{40k+j}\right)\right\},
\end{aligned} \tag{3.5}$$

where the integers (for $t\in\mathbb{Z}, t\neq 0$) a_j, $j=1,2,\ldots,40$ are given by

$$a_j = \frac{f(j)}{5}t^{39-j}\sqrt{5}\sqrt{2^{40-j}}. \tag{3.6}$$

Assigning the explicit values of a_j into equation (3.5) the theorem is proved. □

Remark. The theorem (equation (3.1)) is actually true for any nonzero complex number t, as is readily established by the Principle of Permanence for analytic functions [1]. Technically speaking, however, equation (3.1) is BBP-type only if t is a nonzero integer.

4. A Binary BBP-Type Formula for $\sqrt{5}\log\phi$

Setting $t=1$ in equation (3.1) gives the following corollary.

Corollary.

$$\begin{aligned}
\sqrt{5}\log\phi = \frac{5}{3\cdot 2^{20}}P(1,2^{20},40,(&2^{19},0,2^{18},2^{18},0,0,-2^{16},2^{16},2^{15},0,-2^{14},-2^{14}, \\
&2^{13},0,0,-2^{12},-2^{11},0,-2^{10},0,-2^9,0,-2^8,-2^8,0,0,2^6,-2^6, \\
&-2^5,0,2^4,2^4,-2^3,0,0,2^2,2,0,1,0)).
\end{aligned} \tag{4.1}$$

5. Conclusion

We have derived a new BBP-type formula for a family of logarithms. A binary BBP-type formula for $\sqrt{5}\log\phi$ is obtained as a particular case of the result.

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