PULSATED FIBONACCI RECURRENCES

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Abstract. In this note we define a new type of pulsated Fibonacci sequence. Properties are developed with a successor operator. Some examples are given.

1. Introduction

The motivation for this work goes back to some research of Hall [9], Neumann [14], and Stein [19] on finite models of identities. In order to answer the question of whether every member of a variety is a quasi–group given that every finite member is, Stein [18] found it necessary to examine the intersection of Fibonacci sequences.

Subba Rao [20, 21], Horadam [10], and Shannon [17] investigated the intersection of Fibonacci and Lucas sequences and their generalizations with asymptotic proofs, while Péter Kiss adopted a different approach and supplied many relevant historical references [11]. Atanassov developed coupled recursive sequence which had some obvious intersections [1, 5]. Not considered here are various sequences, such as diatomic sequences, which by their very definitions intersect with many other sequences [14].

In this paper, following previous research (see [2, 3, 4]), a new type of pulsated Fibonacci sequence is developed: 'pulsated' because, in a sense, these sequences expand and contract with regular movements.

2. Definitions

Let a, b, and c be three fixed real numbers. Let us construct the following two recurrent sequences, $\{\alpha_n\}$ and $\{\beta_n\}$ with initial conditions:

$$\alpha_0 = \beta_0 = a, \tag{2.1}$$

$$\alpha_1 = 2b, \tag{2.2}$$

$$\beta_1 = 2c, \tag{2.3}$$

satisfying the combined recurrence relations:

$$\alpha_{2k} = \beta_{2k} = \alpha_{2k-2} + \frac{\alpha_{2k-1} + \beta_{2k-1}}{2}, \tag{2.4}$$

$$\alpha_{2k+1} = \alpha_{2k} + \beta_{2k-1}, \tag{2.5}$$

$$\beta_{2k+1} = \beta_{2k} + \alpha_{2k-1}, \tag{2.6}$$

for every natural number $k \geq 1$. We refer to this pair of intertwined sequences as the $(a; 2b; 2c)$–Pulsated Fibonacci sequence. The first values of the sequence are given in the following table:

Key words and phrases. Fibonacci Sequence, Systems of Recurrences, Successor Operator.

TABLE 1. Initial values for the $(a; 2b; 2c)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$2c$
2	–	$a + b + c$	–
3	$a + b + 3c$	–	$a + 3b + c$
4	–	$2a + 3b + 3c$	–
5	$3a + 6b + 4c$	–	$3a + 4b + 6c$
6	–	$5a + 8b + 8c$	–
7	$8a + 12b + 14c$	–	$8a + 14b + 12c$
8	–	$13a + 21b + 21c$	–

Theorem 2.1. *For every natural number $k \geq 1$, with the elements of the Fibonacci sequence denoted $\{F_n\}$,*

$$\alpha_{2k} = \beta_{2k} = F_{2k-1}a + F_{2k}b + F_{2k}c, \tag{2.7}$$

$$\alpha_{4k-1} = F_{4k-2}a + (F_{4k-1} - 1)b + (F_{4k-1} + 1)c, \tag{2.8}$$

$$\beta_{4k-1} = F_{4k-2}a + (F_{4k-1} + 1)b + (F_{4k-1} - 1)c, \tag{2.9}$$

$$\alpha_{4k+1} = F_{4k}a + (F_{4k+1} + 1)b + (F_{4k+1} - 1)c, \tag{2.10}$$

$$\beta_{4k+1} = F_{4k}a + (F_{4k+1} - 1)b + (F_{4k+1} + 1)c. \tag{2.11}$$

Proof. We proceed by mathematical induction. Obviously, for $k = 1$ the assertion is valid. Let us assume that for some natural number $k \geq 1$, (2.7)–(2.11) hold. For the natural number $k + 1$, first, we check that

$$\alpha_{4k+2} \tag{2.12}$$

$$= \beta_{4k+2} \tag{2.13}$$

$$= \alpha_{4k} + \frac{\alpha_{4k+1} + \beta_{4k+1}}{2} \tag{2.14}$$

$$= F_{4k-1}a + F_{4k}b + F_{4k}c + \frac{F_{4k}a + (F_{4k+1}+1)b + (F_{4k+1}-1)c + F_{4k}a + (F_{4k+1}-1)b + (F_{4k+1}+1)c}{2} \tag{2.15}$$

$$= F_{4k-1}a + F_{4k}b + F_{4k}c + F_{4k}a + F_{4k+1}b + F_{4k+1}c. \tag{2.16}$$

Secondly, we check that

$$\alpha_{4k+1} \tag{2.17}$$

$$= \alpha_{4k+2} + \beta_{4k+1} \tag{2.18}$$

$$= F_{4k+1}a + F_{4k+2}b + F_{4k+2}c + F_{4k}a + (F_{4k+1} - 1)b + (F_{4k+1} + 1)c \tag{2.19}$$

$$= F_{4k+2}a + (F_{4k+3} - 1)b + (F_{4k+3} + 1)c. \tag{2.20}$$

All of the other equalities are checked analogously. □

For example, when $c = -b$, the Pulsated Fibonacci sequence has the form shown in Table 2, while when $c = b$ we obtain Table 3.

TABLE 2. Initial values for the $(a; 2b; -2b)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$-2b$
2	–	a	–
3	$a - 2b$	–	$a + 2b$
4	–	$2a$	–
5	$3a + 2b$	–	$3a - 2b$
6	–	$5a$	–
7	$8a - 2b$	–	$8a + 2b$
8	–	$13a$	–

TABLE 3. Initial values for the $(a; 2b; 2b)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$2b$
2	–	$a + 2b$	–
3	$a + 4b$	–	$a + 4b$
4	–	$2a + 6b$	–
5	$3a + 10b$	–	$3a + 10b$
6	–	$5a + 16b$	–
7	$8a + 26b$	–	$8a + 26b$
8	–	$13a + 42b$	–

Where the coefficients can be easily derived from the result of Theorem 1 by substitution.

3. Discussion

We note that the recursive definitions of α and β may be rewritten in the following form:

$$\alpha_k = \begin{cases} \alpha_{k-2} + \frac{\alpha_{k-1}+\beta_{k-1}}{2} & k \equiv 0 \pmod 2 \\ \alpha_{k-1} + \beta_{k-2} & k \equiv 1 \pmod 2 \end{cases} \tag{3.1}$$

and

$$\beta_k = \begin{cases} \alpha_{k-2} + \frac{\alpha_{k-1}+\beta_{k-1}}{2} & k \equiv 0 \pmod 2 \\ \beta_{k-1} + \alpha_{k-2} & k \equiv 1 \pmod 2 \end{cases} \tag{3.2}$$

This interpretation permits the statement of this problem in terms of the successor operator method introduced by DeTemple and Webb in [7]. Thus, we may define helper sequences

$$w_n = \alpha_{2n}, \tag{3.3}$$

$$x_n = \alpha_{2n+1}, \tag{3.4}$$

$$y_n = \beta_{2n}, \tag{3.5}$$

$$z_n = \beta_{2n+1}. \tag{3.6}$$

This allows us to rewrite (3.1) and (3.2) as

$$w_n = y_n = w_{n-1} + \tfrac{1}{2}x_{n-1} + \tfrac{1}{2}z_{n-1}, \tag{3.7}$$

$$x_n = w_n + z_{n-1}, \tag{3.8}$$

$$z_n = y_n + x_{n-1}. \tag{3.9}$$

Which in terms of the successor operator E gives the following linear system of sequences:

$$\begin{bmatrix} E-1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -E & E & 0 & -1 \\ -1 & -\frac{1}{2} & E & -\frac{1}{2} \\ 0 & -1 & -E & E \end{bmatrix} \begin{bmatrix} w_n \\ x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.10}$$

Thus, the determinant of this system gives the characteristic polynomial of a recurrence relation that annihilates all of the sequences. The determinant is equal to $E(E^3-2E^2-2E+1)$ and hence the sequences $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ all satisfy the third order homogeneous, linear recurrence relation

$$t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}. \tag{3.11}$$

This recurrence (3.11) has eigenvalues $\{-1, \frac{3\pm\sqrt{5}}{2}\}$, and, with initial values of unity yields the 'coupled' sequence $\{1, 1, 1, 3, 7, 19, 49, 129, 337, \ldots\}$ [6]. This sequence appears in the OEIS as A061646, with a variety of combinatorial interpretations [16]. Additionally, the polynomial factors further as $E(E+1)(E^2-3E+1)$. From this factorization the sequence $\{w_n\}$ and $\{y_n\}$ (the even α and β terms) satisfy the second order relation

$$t_n = 3t_{n-1} - t_{n-2}, \tag{3.12}$$

which is also satisfied by alternate terms of the Fibonacci sequence (A001519 and A001906 [16]).

Finally, putting the sequences back together we would expect to need a sixth order recurrence. Instead, we find that both of the original α_n and β_n sequences satisfy the fourth order recurrence

$$t_n = t_{n-1} + t_{n-3} + t_{n-4}. \tag{3.13}$$

This recurrence (3.13) has roots $\{\pm i, \frac{1\pm\sqrt{5}}{2}\}$ and with unit initial values yields the sequence $\{1, 1, 1, 1, 3, 5, 7, 11, 19, 31, 49, 79, 129, \ldots\}$, contained in the OEIS as A126116 [16], of which the couple sequence above is a subsequence. The connections among all these sequence are not surprising since, as is well known, $i^2 = -1$ and $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, and so on.

4. Concluding Comments

In summary then, we have that the given recursive sequences satisfy the following recurrences:

Sequence	Recurrence Relation
α_n and β_n	$t_n = t_{n-1} + t_{n-3} + t_{n-4}$
$w_n = \alpha_{2n} = \beta_{2n} = y_n$	$t_n = 3t_{n-1} - t_{n-2}$
$x_n = \alpha_{2n+1}$ and $z_n = \beta_{2n+1}$	$t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}$

The two sequences discussed in [2, 3] we called 2–Pulsated Fibonacci sequences (from (a;b) and (a;b;c)–types). In [4] they were extended to what were called s–Pulsated Fibonacci sequences, where $s \geq 3$. In future research, it is planned to extend the present

2–Pulsated Fibonacci sequences from $(a; 2b; 2c)$–type, to s–Pulsated Fibonacci sequences from $(a; 2b_1; \ldots, 2b_s)$–type. Other related possibilities for research concern

- conjectures on the number of distinct prime divisors of these sequences [13, 22],
- connections with geometry [6, 8, 12].

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