STEINHAUS TRIANGLES WITH GENERALIZED PASCAL ADDITION

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Abstract. For a given row of 0s and 1s the following row is determined by the sums mod 2 of s consecutive entries each. If this operation is repeated as long as possible then a generalized Steinhaus triangle is obtained which is called balanced if there are as many 0s as 1s. Necessary conditions for the existence of balanced Steinhaus triangles are determined. Constructions are given in most of the cases for odd s and in some cases for even s.

1. Introduction

Consider a sequence $a = (a_1, a_2, \ldots, a_n)$ of length n with $a_i \in \{0, 1\}$. For $n \geq s \geq 2$ the derivative $a' = (a'_1, a'_2, \ldots, a'_{n-s+1})$ is defined by $a'_i = (a_i + a_{i+1} + \ldots + a_{i+s-1}) \bmod 2$ being the addition as in a generalized Pascal triangle [2]. A given sequence a determines a finite sequence $\nabla = \nabla_s(a) = (a, a', a'', \ldots, a^{(t-1)})$ of length $t = \lceil n/(s-1) \rceil$ where $a^{(t-1)}$ is of length less than s and thus has no further derivative. The sequence ∇ can be represented in triangular arrangements, for example, as in Figure 1 for $s = 3$ and $a = 1101010$. These triangles will be

$$
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
&&0&0&1&0&1\\
&&&&1&1&0\\
&&&&&&0
\end{array}
\qquad
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
&0&0&1&0&1&\\
&&1&1&0&&\\
&&&0&&&
\end{array}
\qquad
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
0&0&1&0&1&&\\
1&1&0&&&&\\
0&&&&&&
\end{array}
$$

Figure 1. Triangular arrangements of $\nabla_3(1101010)$.

called (generalized) Steinhaus triangles since they have been introduced for $s = 2$ in [9], that is, using the classical Pascal addition. A Steinhaus triangle is called balanced if there occur as many 0s as 1s in the whole triangle as for example in Figure 1. As Steinhaus did for $s = 2$ we will ask for the existence of balanced Steinhaus triangles for general s and all lengths n.

A first solution for $s = 2$ has been presented in [8] already. A further generalization where $a_i \in \{0, 1, \ldots, m-1\}$ and $a'_i = (a_i + a_{i+1}) \bmod m$ and corresponding references can be found in [3, 4, 5]. For $a'_i = |a_i - a_{i+1}|$ see [1]. Classical Steinhaus triangles ($s = 2$) have been interpreted as incidence matrices of so-called Steinhaus graphs (see for example [6, 7]).

2. Necessary conditions

At first the number $P(n, s)$ of entries in a Steinhaus triangle $\nabla_s(a_1, \ldots, a_n)$ will be determined.

Theorem 2.1. *For $n = (t-1)(s-1) + j$ with $1 \leq j \leq s-1$ and $t \geq 1$ it holds*

$$P(n, s) = \frac{t(n+j)}{2}.$$

Proof. In row r of the t rows there are $(t-r)(s-1)+j$ entries for $1 \le r \le t$. The sum of the t elements of this arithmetic progression having n as first element and j as last element results in the asserted formula. □

Next the values of n will be determined for which $P(n,s)$ is even being necessary for the existence of a balanced Steinhaus triangle.

Theorem 2.2. *The number $P(n,s)$ is even if and only if*

(1) s *odd and*

(1.1) $n \equiv 0 \pmod 2$ *or*

(1.2) $n \equiv s, s+2, \ldots, 2s-3 \pmod{2s-2}$

or

(2) s *even and*

(2.1) $n \equiv 0, 2, \ldots, s-2 \pmod{4s-4}$ *or*

(2.2) $n \equiv 2s-1, 2s+1, \ldots, 3s-3 \pmod{4s-4}$ *or*

(2.3) $n \equiv 3s-2, 3s-1, \ldots, 4s-5 \pmod{4s-4}$.

Proof. Using $n = (t-1)(s-1)+j$, $1 \le j \le s-1$, we have

$$P = P(n,s) = \frac{t((t-1)(s-1)+2j)}{2}.$$

Let s be odd. If $t = 2i+1$ then $P = (2i+1)(i(s-1)+j)$ which is even if and only if j is even, that is, $j = 2, 4, \ldots, s-1$. Since $n = i(2s-2)+j$ it follows $n \equiv 2, 4, \ldots s-1 \pmod{2s-2}$. If $t = 2i$ then $P = i((2i-1)(s-1)+2j)$ which is always even, that is, for $j = 1, 2, \ldots s-1$. Since $n = (2i-1)((s-1)+j) = (i-1)(2s-2)+s+j-1$ it follows $n \equiv s, s+1, \ldots, 2s-2 \pmod{2s-2}$. Together both sets of values of n correspond to (1.1) and (1.2).

Let s be even. If $t = 4i+1$ then $P = (4i+1)(2i(s-1)+j)$ is even if and only if j is even, that is, $j = 2, 4, \ldots, s-2$. From $n = i(4s-4)+j$ we obtain $n \equiv 2, 4, \ldots, s-2 \pmod{4s-4}$. If $t = 4i+3$ then $P = (4i+3)((2i+1)(s-1)+j)$ is even if and only if j is odd, that is, $j = 1, 3, \ldots, s-1$. From $n = (4i+2)(s-1)+j = i(4s-4)+2s+j-2$ we have $n \equiv 2s-1, 2s+1, \ldots, 3s-3 \pmod{4s-4}$. If $t = 4i+2$ then $P = (2i+1)((4i+1)(s-1)+2j)$ which is always odd. If $t = 4i$ then $P = 2i((4i-1)(s-1)+2j)$ which is always even, that is, for $j = 1, 2, \ldots, s-1$. With $n = (4i-1)(s-1)+j = (i-1)(4s-4)+3s+j-3$ we obtain $n \equiv 3s-2, 3s-1, \ldots, 4s-4 \pmod{4s-4}$. Together these three sets of values of n correspond to (2.1) to (2.3). □

Theorem 2.3. *If $s \equiv 1 \pmod 4$ and $n = s$ or if $s \equiv 0 \pmod 2$, $s > 2$, and $n = 3s$ then balanced Steinhaus triangles do not exist although $P(n,s)$ is even.*

Proof. If $s \equiv 1 \pmod 4$ and $n = s$ then $P(n,s) = s+1 \equiv 2 \pmod 4$ follows since we have $a = (a_1, a_2, \ldots, a_n)$ and $a' = (a'_1)$ with $a'_1 = (a_1 + a_2 + \ldots + a_n) \bmod 2$. Now the total number of 1s is $a_1 + a_2 + \ldots + a_n + a'_1 \equiv 2(a_1 + a_2 + \ldots + a_n) \equiv 0 \pmod 2$ so that the triangle cannot be balanced since $P(n,s)/2 \equiv 1 \pmod 2$.

If $s \equiv 0 \pmod 2$, $s > 2$, and $n = 3s$ then $t = \lceil 3s/(s-1) \rceil = 3 + \lceil 3/(s-1) \rceil = 4$ and $j = 3$. This implies $P(n,s)/2 = 4(3s+3)/4 = 3(s+1) \equiv 1 \pmod 2$. Then the triangle cannot be balanced if we prove that the number of 1s is always even, that is, the total sum of the entries is even.

This is the case if the number of entries in which each a_i occurs an odd number of times is even, that is, in a triangle with only one 1 in the first row the total number of 1s is always even. We start with a triangle with 0s only (see italic values in Figure 2 for $s = 6$), thus we have an even number of 1s. Then we shift the single 1 from right to left through the first row.

$$\begin{array}{c}
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}\ \underline{\mathbf{1}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}\ \underline{1}\ 1\ 1\ 1\ 1\ \mathbf{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0}\ \underline{1}\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \mathbf{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0}\ \underline{1}\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ \mathbf{1}\ 0\ 0\ 0
\end{array}$$

FIGURE 2. Example for $s = 6$.

The corresponding derivatives are shifted simultaneously. In the first step the four underlined 1s are shifted into the triangle and at the same time the four bold 0s at the left border leave the triangle. Due to symmetry the four underlined 1s correspond to the four bold 1s. Figures 3 and 4 show a, a', a'', and a''' for general $s \equiv 0$ and $s \equiv 2 \pmod 4$ such that the four bold 1s and the four bold 0s are in one column each, that is, in columns $i = 0$ and $i = 3s$. In the

$$\begin{array}{ll}
i & = 0\ 1 \qquad\quad \dots \qquad\quad s \quad \dots \quad 2s \qquad\qquad \dots \qquad\qquad 3s \\[6pt]
a & = \mathbf{0}|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 0 \dots 0\ 0|0\ 0\ 0|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|\mathbf{1} \\
a' & = \mathbf{0}|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 0 \dots 0\ 0|1\ 1\ 1|1\ 1\ 1\ 1 \dots 1\ 1\ 1\ 1|\mathbf{1} \\
a'' & = \mathbf{0}|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 1 \dots 0\ 1|0\ 1\ 0|1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 0|\mathbf{1} \\
a''' & = \mathbf{0}|0\ 0\ 1\ 1 \dots 0\ 0\ 1\ 1|0\ 0 \dots 0\ 0|0\ 0\ 1|1\ 0\ 0\ 1 \dots 1\ 0\ 0\ 1|\mathbf{1} \\[6pt]
\sigma & = 0|0\ 0\ 1\ 1 \dots 0\ 0\ 1\ 1|0\ 1 \dots 0\ 1|1\ 0\ 0|1\ 1\ 0\ 0 \dots 1\ 1\ 0\ 0|0
\end{array}$$

FIGURE 3. $s \equiv 0 \pmod 4$.

$$\begin{array}{ll}
i & = 0\ 1 \qquad\qquad \dots \qquad\qquad s \quad \dots \quad 2s \qquad\qquad \dots \qquad\qquad 3s \\[6pt]
a & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 0 \dots 0\ 0|0|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|\mathbf{1} \\
a' & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 0 \dots 0\ 0|1|1\ 1\ 1\ 1 \dots 1\ 1\ 1\ 1|\mathbf{1} \\
a'' & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0|0\ 1 \dots 0\ 1|0|1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 0|\mathbf{1} \\
a''' & = \mathbf{0}\ 0\ 0|1\ 1\ 0\ 0 \dots 1\ 1\ 0\ 0|1\ 1 \dots 1\ 1|1|1\ 0\ 0\ 1 \dots 1\ 0\ 0\ 1|\mathbf{1} \\[6pt]
\sigma & = 0\ 0\ 0|1\ 1\ 0\ 0 \dots 1\ 1\ 0\ 0|1\ 0 \dots 1\ 0|0|1\ 1\ 0\ 0 \dots 1\ 1\ 0\ 0|0
\end{array}$$

FIGURE 4. $s \equiv 2 \pmod 4$.

following steps the four entries in column $i = 3s - j$ are shifted into the triangle and at the same time the four entries in column $i = j$ leave the triangle for $j = 1, \dots, 3s - 1$. The total number of 1s in the triangle remains even if the sum of the entering and leaving entries is even in every step. This can be checked in Figures 3 and 4 since for the sums σ_i of the columns modulo 2 it holds $\sigma_j = \sigma_{3s-j}$ where $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{3s})$. □

Theorem 2.3 shows that some of the conditions for n in Theorem 2.2 are not sufficient. For $s = 2^u$ we conjecture in addition to Theorem 2.3 that there exist no balanced Steinhaus triangles for $n = s(4s^i - 1)$, $i = 1, 2, \dots$, although $P(n, s)$ is even. This has been checked by computer for $s, n \le 100000$. Moreover, we conjecture that balanced Steinhaus triangles exist in all other cases where $P(n, s)$ is even.

3. Odd values of s

For one half of the odd values of s the existence of balanced triangles can be proved completely.

Theorem 3.1. *For $s \equiv 3 \pmod 4$ balanced Steinhaus triangles exist for all n fulfilling the necessary conditions (1.1) or (1.2) in Theorem 2.2.*

Proof. If $a = 0101\ldots$ then $a' = 1010\ldots$, $a'' = 0101\ldots$ and thus every second row has the same pattern. For n even the triangle is balanced since all rows are of even length. For n odd all rows are of odd length and all pairs of consecutive rows are balanced. Then the triangle is balanced if the number t of rows is even which is the case for the odd values n in (1.2), see the proof of Theorem 2.2. □

For the other half of the odd values of s one residue class remains open up to some small examples.

Theorem 3.2. *For $s \equiv 1 \pmod 4$ balanced Steinhaus triangles exist for all $n \equiv 0 \pmod 2$ and all $n \equiv s+2, s+4, \ldots, 2s-3 \pmod{2s-2}$.*

Proof. If $a = 0101\ldots$ it follows $a' = 0101\ldots$ so that every row has the same pattern and the triangle is balanced for all even values of n since then all rows are of even length.

For odd n we choose $a = 00(01)1$ to obtain $a' = 10(01)1$ and $a'' = 00(01)1$ where the part in brackets can be repeated arbitrarily often. It follows that all pairs of consecutive rows are balanced. Thus for $t = 2i + 2$ the triangle is balanced, that is, for $n = (t-1)(s-1) + j = i(2s-2) + s - 1 + j$. With $j = 3, 5, \ldots, s-2$ we have obtained a solution for $n \equiv s+2, s+4, \ldots, 2s-3 \pmod{2s-2}$. □

Note that the construction in the preceding proof is not possible for $j = 1$, that is, for $n \equiv s \pmod{2s-2}$. Due to Theorems 2.2 and 2.3 for $s \equiv 1 \pmod 4$, $n = s + i(2s-2)$, $i = 1, 2, \ldots$, balanced triangles may be possible. The following theorem guarantees such balanced triangles for some residue classes of n by general construction.

Theorem 3.3. *For $s \equiv 1 \pmod 4$ and $s \geq 1 + 2^c$, $c \geq 1$, balanced Steinhaus triangles exist for $n \equiv s + (s-1)2^c \pmod{(2s-2)2^c}$.*

Proof. We define $2^c \times 2^c$ matrices M_c, $c \geq 1$, recursively by

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{c+1} = \left(\begin{array}{c|c} M_c & M_c \\ \hline 0\ 1\ \ldots\ 0\ 1 & \\ \ldots & M_c \\ 0\ 1\ \ldots\ 0\ 1 & \end{array}\right).$$

For example,

$$M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If every row of M_c is augmented by an infinite sequence 0101... to the right hand side then for $s \equiv 1 \pmod 4$ and $s \geq 2^c$ the first row is the derivative of the last row and each of all other rows is the derivative of the preceding row. This can be seen by induction on c using that each but the last row of M_c has an even number of 1s for $c \geq 2$.

Now let P_c be the $2^{c+1} \times (1+2^c)$ matrix consisting of the $1+2^c$ rightmost columns of M_{c+1}. Note that the first column of P_c consists of 2^c consecutive 0s followed by 2^c consecutive 1s. Let Q_c be the $2^{c+1} \times 2^c$ matrix obtained from the 2^c rightmost columns of M_{c+1} by horizontal reflection and taking the complement. Then we define the trapezoid-like scheme T_c by

$$T_c = \left(\begin{array}{c|ccccc|c} & 0 & 1 & \ldots & 0 & 1 & \\ P_c & & & \ldots & & & Q_c \\ & 0 & 1 & \ldots & 0 & 1 & \end{array} \right)$$

where each row has $(s-1)/2$ pairs 01 less that its preceding one in the central part. Note that the first row of T_c has the same pattern (only different numbers of pairs 01 in the central part) as the derivative of the last row and each of all other rows is the derivative of its preceding one. Moreover, T_c is balanced since the first column is balanced and each row of the rest of T_c is balanced, since the rest is the complement of its mirror image. For example, see Figure 5.

$$T_2 = \left(\begin{array}{ccccc|ccccc|cccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right)$$

FIGURE 5. T_2.

For $s \geq 1+2^{c+1}$ we choose from T_c the last $1+2^{c-1}$ rows beginning with 0 followed by the first 2^{c-1} rows beginning with 1 (see Figure 6 for $s = 9$ and 13 with the first and last horizontal line of Figure 5). If there are $(s-(1+2^{c+1}))/2$ pairs 01 in the last of these chosen rows then

$$\begin{array}{l} \hline 0\,0\,0\,0\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,1\,1\,1 \\ 0\,0\,1\,1\,0 \mid 0\,1\,0\,1\,0\,1\,0\,1 \mid 1\,0\,0\,1 \\ 0\,0\,1\,0\,0 \mid 0\,1\,0\,1 \mid 1\,1\,0\,1 \\ 1\,1\,0\,1\,0 \mid 1\,0\,1\,0 \\ \hline 1\,1\,0\,1\,0 \\ 1 \end{array} \qquad \begin{array}{l} \hline 0\,0\,0\,0\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,1\,1\,1 \\ 0\,0\,1\,1\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,0\,0\,1 \\ 0\,0\,1\,0\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,1\,0\,1 \\ 1\,1\,0\,1\,0 \mid 0\,1\,0\,1\,0\,1\,0\,1 \mid 1\,0\,1\,0 \\ 1\,0\,0\,0\,0 \mid 1\,1\,1\,1 \\ \hline 1 \end{array}$$

$$\begin{array}{l} \hline 0\,0\,0\,0\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,1\,1\,1 \\ 0\,0\,1\,1\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,0\,0\,1 \\ 0\,0\,1\,0\,0 \mid 0\,1 \;\ldots\; 0\,1 \mid 1\,1\,0\,1 \\ 1\,1\,0\,1\,0 \mid 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1 \mid 1\,0\,1\,0 \\ 1\,0\,0\,0\,0 \mid 0\,1\,0\,1 \mid 1\,1\,1\,1 \\ \hline 1 \end{array}$$

FIGURE 6. $(s, n) = (5, 21)$, $(9, 41)$, and $(13, 61)$.

it has s entries and its derivative is a single 1. Thus we have constructed a Steinhaus triangle being balanced since the first column has as many 1s as 0s. If on top of this triangle $i2^{c+1}$ rows are added which are the cyclically preceding rows of T_c then we obtain balanced triangles for

$$\begin{aligned} n &= 1 + (s-1)(2^{c-1} + 2^{c-1} + 1 + i2^{c+1}) \\ &= s + (2^{c-1} + i2^c)(2s-2) \\ &\equiv s + (s-1)2^c \pmod{(2s-2)2^c}. \end{aligned}$$

For $1 + 2^c \le s \le 2^{c+1}$ we can proceed similar to the preceding case. From T_c we now choose the last $1+2^{c-1}$ rows beginning with 0 followed by the first $2^{c-1}-1$ rows beginning with 1 (see Figure 6 for $s = 5$ with the first and second horizontal line of Figure 5). If there are $s-1-2^c$ pairs 01 in the last of the chosen rows then it has $1 + 2^{c+1} + 2(s-1-2^c) = 2s-1$ entries and its derivative has length s and starts with 1. The rest of this row is the complement of its mirror image and thus it is balanced and has an even number of 1s. Then the derivative of this row is a single 1 and we have balanced triangles for

$$n = 1 + (s-1)(1 + 2^{c-1} - 1 + 2^{c-1} + 1 + i2^{c+1})$$

as above. □

For $s \equiv 1 \pmod 4$ and c such that $1 + 2^c \le s \le 2^{c+1}$ by Theorems 2.2, 2.3, 3.2, and 3.3 only the cases $n \equiv s \pmod{(2s-2)2^c}$ remain open. For $s = 5$ and $s = 9$ balanced Steinhaus triangles exist in these cases:

$$\begin{aligned} s = 5:\quad & n \equiv 5 \pmod{64},\ n \ne 5: && a = 11111111110(01), \\ & n \equiv 37 \pmod{64}: && a = 00100100010(01), \\ s = 9:\quad & n \equiv 9 \pmod{256},\ n \ne 9: && \\ & \qquad a = 010111011101000100(01)11011101000100010, && \\ & n \equiv 137 \pmod{256}: && \\ & \qquad a = 001010110010101100(01)1101010110010101, && \end{aligned}$$

where the part between the brackets is repeated appropriately.

4. Even values of s

The case $s = 2$ is solved in [8]. Moreover, for $s \equiv 0 \pmod 2$ we so far have found balanced Steinhaus triangles for some small values of s only:

$$\begin{aligned} s = 4:\quad & n \equiv 0 \pmod{24}: && a = 000100(001111), \\ & n \equiv 2 \pmod{12}: && a = 1000(011110)1000, \\ & n \equiv 7 \pmod{12}: && a = 100(010100)1000, \\ & n \equiv 9 \pmod{12}: && a = 1000(100010)1000, \\ & n \equiv 10 \pmod{12}: && a = 1010(010001), \\ & n \equiv 11 \pmod{12}: && a = 01010(010001), \end{aligned}$$

For $s = 4$ the existence of balanced Steinhaus triangles remains open for $n \equiv 12 \pmod{24}$, $n \ne 12$.

$$
\begin{array}{lll}
s = 6: & n \equiv 0 \pmod{20}: & a = 0011100010(0010000010),\\
& n \equiv 2 \pmod{20},\ n \neq 2: & a = 10111000001(0010000010),\\
& n \equiv 4 \pmod{20}: & a = 0011(0010000010),\\
& n \equiv 11 \pmod{20}: & a = 10111000001(0001000001),\\
& n \equiv 13 \pmod{20}: & a = 0111010000010(0010000010),\\
& n \equiv 15 \pmod{20}: & a = 00111(0010000010),\\
& n \equiv 16 \pmod{20}: & a = 100110(0010000010),\\
& n \equiv 17 \pmod{20}: & a = 1000111(0010000010),\\
& n \equiv 19 \pmod{20}: & a = 100100110(0010000010),\\
& n \equiv 38, 58, 78 \pmod{80}: & a = 000101101011100010(0010000010),\\
& n \equiv 98 \pmod{160}: & a = 110001101010101111(0010000010),
\end{array}
$$

For $s = 6$ the case $n \equiv 18 \pmod{160}$, $n \neq 18$, remains open.

5. Remarks

Summarizing, the existence of balanced Steinhaus triangles with generalized Pascal addition in the case of odd n remains open only for $s \equiv 1 \pmod 4$ and large values of n with $n \equiv s \pmod{2s-2}$. In the case of n even only examples for small values of s and some classes of n are known.

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