GENERALIZING ZECKENDORF'S THEOREM: THE KENTUCKY SEQUENCE

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Abstract. By Zeckendorf's theorem, an equivalent definition of the Fibonacci sequence (appropriately normalized) is that it is the unique sequence of increasing integers such that every positive number can be written uniquely as a sum of non-adjacent elements; this is called a legal decomposition. Previous work examined the distribution of the number of summands, and the spacings between them, in legal decompositions arising from the Fibonacci numbers and other linear recurrence relations with non-negative integral coefficients. These results were restricted to the case where the first term in the defining recurrence was positive. We study a generalization of the Fibonacci sequence with a simple notion of legality which leads to a recurrence where the first term vanishes. We again have unique legal decompositions, Gaussian behavior in the number of summands, and geometric decay in the distribution of gaps.

Contents

1. Introduction

One of the standard definitions of the Fibonacci numbers $\{F_n\}$ is that it is the unique sequence satisfying the recurrence $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_1 = 1$, $F_2 =$

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2. An interesting and equivalent definition is that it is the unique increasing sequence of positive integers such that every positive number can be written uniquely as a sum of non-adjacent elements of the sequence.[1] This equivalence is known as Zeckendorf's theorem [27], and frequently one says every number has a unique legal decomposition as a sum of non-adjacent Fibonacci numbers.

Past research regarding generalized Zeckendorf decompositions have involved sequences $\{G_n\}$ where the recurrence relation coefficients are non-negative integers, with the additional restriction being that the first and last terms are positive.[2] See for instance [22], where the authors call these **Positive Linear Recurrence** (PLR) Sequences. In this setting, much is known about the properties of the summands including that the distribution of the number of summands converges to a Gaussian, [9, 23]. There have also been recent results about gaps between summands, including a proof that the distribution of the longest gap converges to the same distribution one sees when looking at the longest run of heads in tosses of a biased coin, see [2, 3, 5]. There is a large set of literature addressing generalized Zeckendorf decompositions, these include[1, 8, 10, 11, 12, 13, 14, 15, 16, 17, 25, 26] among others.

However, all of these results only hold for PLR Sequences. In this paper, we extend the results on Gaussian behavior and average gap measure to recurrences that cannot be handled by existing techniques. To that end, we study a sequence arising from a notion of a legal decomposition whose recurrence has first term equal to zero.[3] While our sequence fits into the framework of an f-decomposition introduced in [9], their arguments only suffice to show that our decomposition rule leads to unique decompositions. The techniques in [9] do not address the distribution of the number of summands nor the behavior of the gaps between the summands for our particular sequence. We address these questions completely in Theorems 1.5 and 1.6, respectively.

We now describe our object of study. We can view the decomposition rule corresponding to the Fibonacci sequence by saying the sequence is divided into bins of length 1, and (i) we can use at most one element from a bin at most one time, and (ii) we cannot choose elements from adjacent bins. This suggests a natural extension where the bins now contain b elements and any two summands of a decomposition (i) cannot be members of the same bin and (ii) must be at least s bins away from each other. We call this the (s, b)**-Generacci sequence** (see Definition 5.2) and the Fibonacci numbers are the $(1, 1)$-Generacci sequence. In this paper we consider the case $s = 1, b = 2$. We give this special sequence a name: the **Kentucky** sequence, after the home state of one of our authors. Although we expect our results to extend in full generality, we have found that new techniques are needed for the two parameter family. See Section 5 for more details on the general case.

Definition 1.1. *Let an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ be given and partition the elements into bins*

$$\mathcal{B}_k := \{a_{2k-1}, a_{2k}\}$$

for $k \geq 1$. We declare a decomposition of an integer

$$m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$$

where $\ell_1 < \ell_2 < \cdots < \ell_k$ and $\{a_{\ell_j}, a_{\ell_{j+1}}\} \not\subset \mathcal{B}_i \cup \mathcal{B}_{i-1}$ for any i, j to be a Kentucky legal decomposition.

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[1] If we started the Fibonacci numbers with a zero, or with two ones, we would lose uniqueness of decompositions.

[2] Thus $G_{n+1} = c_1 G_n + \cdots + c_L G_{n-(L-1)}$ with $c_1 c_L > 0$ and $c_i \geq 0$.

[3] Thus $G_{n+1} = c_1 G_n + c_2 G_{n-1} + \cdots + C_L G_{n-(L-1)}$ with $c_1 = 0$ and $c_i \geq 0$.

This says that we cannot decompose a number using more than one summand from the same bin or two summands from adjacent bins.

Definition 1.2. *An increasing sequence of positive integers* $\{a_i\}_{i=1}^{\infty}$ *is called a* Kentucky sequence *if every* a_i $(i \geq 1)$ *is the smallest positive integer that does not have a Kentucky legal decomposition using the elements* $\{a_1, \dots, a_{i-1}\}$.

From the definition of a Kentucky legal decomposition, the reader can see that the first five terms of the sequence must be $\{1, 2, 3, 4, 5\}$. We have $a_6 \neq 6$ as $6 = a_1 + a_5 = 1 + 5$ is a Kentucky legal decomposition. In the same way we find $a_6 \neq 7$, and this is the largest integer that could be legally decomposed using the first five entries in the sequence. Thus we must have $a_6 = 8$. Continuing we have the first few terms of the Kentucky sequence:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \underbrace{11,\ 16}_{\mathcal{B}_4},\ \underbrace{21,\ 32}_{\mathcal{B}_5},\ \underbrace{43,\ 64}_{\mathcal{B}_6},\ \underbrace{85,\ 128}_{\mathcal{B}_7},\ \underbrace{171,\ 256}_{\mathcal{B}_8},\ \dots$$

We have a nice closed form expression for the elements of this sequence.

Theorem 1.3. *If* $\{a_n\}$ *is the Kentucky sequence, then*

$$a_{n+1} \;=\; a_{n-1} + 2a_{n-3},\ a_1 \;=\; 1,\ a_2 \;=\; 2,\ a_3 \;=\; 3,\ a_4 \;=\; 4,$$

which implies

$$a_{2n} \;=\; 2^n \quad \text{and} \quad a_{2n-1} \;=\; \frac{1}{3}\left(2^{n+1} + (-1)^n\right).$$

This is not a PLR Sequence as the leading coefficient (that of a_n) is zero, and this sequence falls outside the scope of many of the previous techniques. We prove the following theorems concerning the Kentucky Sequence.

Theorem 1.4 (Uniqueness of Decompositions)**.** *Every positive integer can be written uniquely as a sum of distinct terms from the Kentucky sequence where no two summands are in the same bin and no two summands belong to consecutive bins in the sequence.*

The above follows immediately from the work in [9] on f-decompositions. In Theorem 1.3 of [9] take $f(n) = 3$ if n is even and $f(n) = 2$ otherwise. For completeness we give an elementary proof in Appendix A. generalize the results on Gaussian behavior for the summands to this case.

Theorem 1.5 (Gaussian Behavior of Summands)**.** *Let the random variable* Y_n *denote the number of summands in the (unique) Kentucky decomposition of an integer picked at random from* $[0, a_{2n+1})$ *with uniform probability.*[4] *Normalize* Y_n *to* $Y_n' = (Y_n - \mu_n)/\sigma_n$*, where* μ_n *and* σ_n *are the mean and variance of* Y_n *respectively, which satisfy*

$$\begin{aligned}\mu_n &= \frac{n}{3} + \frac{2}{9} + O\left(\frac{n}{2^n}\right)\\ \sigma_n^2 &= \frac{2n}{27} + \frac{8}{81} + O\left(\frac{n^2}{2^n}\right).\end{aligned}$$

Then Y_n' *converges in distribution to the standard normal distribution as* $n \to \infty$.

[4]Using the methods of [4], these results can be extended to hold almost surely for a sufficiently large subinterval of $[0, a_{2n+1})$.

Our final results concern the behavior of gaps between summands. For the legal decomposition

$$m \ = \ a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k} \quad \text{with} \quad \ell_1 \ < \ \ell_2 \ < \ \cdots \ < \ \ell_k$$

and $m \in [0, a_{2n+1})$, we define the set of gaps as follows:

$$\text{Gaps}_n(m) \ := \ \{\ell_2 - \ell_1, \ell_3 - \ell_2, \ldots, \ell_k - \ell_{k-1}\}.$$

Notice we do not include the wait to the first summand, $\ell_1 - 0$, as a gap. We could include this if we wish; one additional gap will not affect the limiting behavior.

In the theorem below we consider all the gaps between summands in Kentucky legal decompositions of all $m \in [0, a_{2n+1})$. We let $P_n(g)$ be the fraction of all these gaps that are of length g (i.e., the probability of a gap of length g among Kentucky legal decompositions of $m \in [0, a_{2n+1})$). For example, notice $m = a_1 + a_{11} + a_{15} + a_{22} + a_{26}$ contributes two gaps of length 4, one gap of length 7 and one gap of length 10.

Theorem 1.6 (Average Gap Measure)**.** *For $P_n(g)$ as defined above, the limit $P(g) := \lim_{n\to\infty} P_n(g)$ exists, and*

$$P(0) \ = \ P(1) \ = \ P(2) \ = \ 0, \quad P(3) \ = \ 1/8,$$

and for $g \geq 4$ we have

$$P(g) \ = \ \begin{cases} 2^{-j} & \text{if } g = 2j \\ \frac{3}{4}\, 2^{-j} & \text{if } g = 2j+1. \end{cases}$$

In §2 we derive the recurrence relation and explicit closed form expressions for the terms of the Kentucky sequence, as well as a useful generating function for the number of summands in decompositions. We then prove Theorem 1.5 on Gaussian behavior in §3, and Theorem 1.6 on the distribution of the gaps in §4. We end with some concluding remarks and directions for future research in §5.

2. Recurrence Relations and Generating functions

In the analysis below we constantly use the fact that every positive integer has a unique Kentucky legal decomposition; see [9] or Appendix A for proofs.

2.1. Recurrence Relations.

Proposition 2.1. *For the Kentucky sequence, $a_n = n$ for $1 \leq n \leq 5$ and for any $n \geq 5$ we have $a_n = a_{n-2} + 2a_{n-4}$. Further for $n \geq 1$ we have*

$$a_{2n} \ = \ 2^n \quad \text{and} \quad a_{2n-1} \ = \ \frac{1}{3}\left(2^{n+1} + (-1)^n\right). \tag{2.1}$$

Proof. We recall that the integers a_{2n+1} and a_{2n} in the Kentucky sequence are elements of the sequence as they are the smallest integers that cannot be legally decomposed using the members of $\{a_1, a_2, \ldots, a_{2n}\}$ or $\{a_1, a_2, \ldots, a_{2n-1}\}$ respectively:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \underbrace{11,\ 16}_{\mathcal{B}_4},\ \underbrace{21,\ 32}_{\mathcal{B}_5},\ \underbrace{43,\ 64}_{\mathcal{B}_6},\ \cdots,\ \underbrace{a_{2n-3},\ a_{2n-2}}_{\mathcal{B}_{n-1}},\ \underbrace{a_{2n-1},\ a_{2n}}_{\mathcal{B}_n}.$$

As a_{2n} is the largest entry in the bin $\mathcal{B}_n$, it is one more than the largest number we can legally decompose, and thus

$$a_{2n} \ = \ a_{2n-1} + a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1$$

where $a_j = a_2$ if n is odd and $a_j = a_4$ if n is even. By construction of the Kentucky sequence we have $a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1 = a_{2(n-2)+1} = a_{2n-3}$. Thus

$$a_{2n} \;=\; a_{2n-1} + a_{2n-3}. \tag{2.2}$$

Similarly a_{2n+1} is the smallest entry in bin $\mathcal{B}_{n+1}$, so

$$a_{2n+1} \;=\; a_{2n} + a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1$$

where $a_j = a_2$ if n is odd and $a_j = a_4$ if n is even. Thus

$$a_{2n+1} \;=\; a_{2n} + a_{2n-3}. \tag{2.3}$$

Substituting (2.2) into (2.3) yields

$$a_{2n+1} \;=\; a_{2n-1} + 2a_{2n-3}, \tag{2.4}$$

and thus for $m \geq 5$ odd we have $a_m = a_{m-2} + 2a_{m-4}$.

Now using (2.4) in (2.2), we have

$$a_{2n} \;=\; a_{2n-1} + a_{2n-3} \;=\; a_{2n-3} + 2a_{2n-5} + a_{2n-3} \;=\; 2(a_{2n-3} + a_{2n-5}).$$

Shifting the index in (2.2) gives

$$a_{2n} \;=\; 2a_{2n-2}. \tag{2.5}$$

Since $a_2 = 2$ and $a_4 = 4$, together with (2.5) we now have $a_{2n} = 2^n$ for all $n \geq 1$. A few algebraic steps then confirm $a_m = a_{m-2} + 2a_{m-4}$ for $m \geq 6$ even.

Finally, we prove that $a_{2n-1} = \frac{1}{3}(2^{n+1} + (-1)^n)$ for $n \geq 1$ by induction. The base case is immediate as $a_1 = 1$ and $\frac{1}{3}(2^{1+1} + (-1)^1) = \frac{1}{3}(4-1) = 1$. Assume for some $N \geq 1$, $a_{2N-1} = \frac{1}{3}(2^{N+1} + (-1)^N)$. By (2.4), we have

$$\begin{aligned}
a_{2(N+1)-1} &= a_{2N+1} \\
&= a_{2N-1} + 2a_{2N-3} \\
&= \frac{1}{3}(2^{N+1} + (-1)^N) + (2)\left(\frac{1}{3}\right)(2^{N-1+1} + (-1)^{N-1}) \\
&= \frac{1}{3}(2^{N+1} + (-1)^N + 2^{N+1} + (-1)^{N-1} + (-1)^{N-1}) \\
&= \frac{1}{3}(2^{N+2} + (-1)^{N+1}),
\end{aligned}$$

and thus for all $n \geq 1$ we have $a_{2n-1} = \frac{1}{3}(2^{n+1} + (-1)^n)$. □

2.2. **Counting Integers With Exactly k Summands.** In [18], Koloğlu, Kopp, Miller and Wang introduced a very useful combinatorial perspective to attack Zeckendorf decomposition problems. While many previous authors attacked related problems through continued fractions or Markov chains, they instead partitioned the $m \in [F_n, F_{n+1})$ into sets based on the number of summands in their Zeckendorf decomposition. We employ a similar technique here, which when combined with identities about Fibonacci polynomials allows us to easily obtain Gaussian behavior.

Let $p_{n,k}$ denote the number of $m \in [0, a_{2n+1})$ whose legal decomposition contains exactly k summands where $k \geq 0$. We have $p_{n,0} = 1$ for $n \geq 0$, $p_{0,k} = 0$ for $k > 0$, $p_{1,1} = 2$, and $p_{n,k} = 0$

if $k > \lfloor \frac{n+1}{2} \rfloor$. Also, by definition,

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} p_{n,k} \;=\; a_{2n+1},$$

and we have the following recurrence.

Proposition 2.2. *For $p_{n,k}$ as above, we have*

$$p_{n,k} \;=\; 2p_{n-2,k-1} + p_{n-1,k}$$

for $n \geq 2$ and $k \leq \lfloor \frac{n+1}{2} \rfloor$.

Proof. We partition the Kentucky legal decompositions of all $m \in [0, a_{2n+1})$ into two sets, those that have a summand from bin $\mathcal{B}_n$ and those that do not.

If we have a legal decomposition $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ with $a_{\ell_k} \in \mathcal{B}_n$, then $a_{\ell_{k-1}} \leq a_{2(n-2)}$ and there are two choices for a_{ℓ_k}. The number of legal decompositions of the form $a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_{k-1}}$ with $a_{\ell_{k-1}} \leq a_{2(n-2)}$ is $p_{n-2,k-1}$. Note the answer is independent of which value $a_{\ell_k} \in \mathcal{B}_n$ we have. Thus the number of legal decompositions of m containing exactly k summands with largest summand in bin $\mathcal{B}_n$ is $2p_{n-2,k-1}$.

If $m \in [0, a_{2n+1})$ does not have a summand from bin $\mathcal{B}_n$ in its decomposition, then $m \in [0, a_{2n-1})$, and by definition the number of such m with exactly k summands in a legal decomposition is $p_{n-1,k}$.

Combining these two cases yields

$$p_{n,k} \;=\; 2p_{n-2,k-1} + p_{n-1,k},$$

completing the proof. □

This recurrence relation allows us to compute a closed-form expression for $F(x, y)$, the generating function of the $p_{n,k}$'s.

Proposition 2.3. *Let*

$$F(x, y) \;:=\; \sum_{n,k \geq 0} p_{n,k} x^n y^k$$

be the generating function of the $p_{n,k}$'s arising from Kentucky legal decompositions. Then

$$F(x, y) \;=\; \frac{1 + 2xy}{1 - x - 2x^2 y}. \tag{2.6}$$

Proof. Noting that $p_{n,k} = 0$ if either $n < 0$ or $k < 0$, using explicit values of $p_{n,k}$ and the recurrence relation from Proposition 2.2, after some straightforward algebra we obtain

$$F(x, y) \;=\; 2x^2 y F(x, y) + x F(x, y) + 2xy + 1.$$

From this, (2.6) follows. □

While the combinatorial vantage of [18] has been fruitfully applied to a variety of recurrences (see [22, 23]), their proof of Gaussianity does not generalize. The reason is that for the Fibonacci numbers (which are also the $(1, 1)$-Generacci numbers) we have an explicit, closed form expression for the corresponding $p_{n,k}$'s, which greatly facilitates the analysis. Fortunately for us a similar closed form expression exists for Kentucky decompositions.

Proposition 2.4. *Let $p_{n,k}$ be the number of integers in $[0, a_{2n+1})$ that have exactly k summands in their Kentucky legal decomposition. For all $k \geq 1$ and $n \geq 1 + 2(k-1)$, we have*

$$p_{n,k} \;=\; 2^k \binom{n-(k-1)}{k}.$$

Proof. We are counting decompositions of the form $a'_{\ell_1} + \cdots + a'_{\ell_k}$ where $a'_{\ell_i} \in \mathcal{B}_{\ell_i} = \{a_{2\ell_i - 1}, a_{2\ell_i}\}$ and $\ell_i \leq n$. Define $x_1 := \ell_1 - 1$ and $x_{k+1} := n - \ell_k$. For $2 \leq i \leq k$, define $x_i := \ell_i - \ell_{i-1} - 1$. We have

$$x_1 + 1 + x_2 + 1 + x_3 + 1 + \cdots + x_k + 1 + x_{k+1} \;=\; n.$$

We change variables to rewrite the above. Essentially what we are doing is replacing the x's with new variables to reduce our Diophantine equation to a standard form that has been well-studied. As we have a legal decomposition, our bins must be separated by at least one and thus $x_i \geq 1$ for $2 \leq i \leq k-1$ and $x_1, x_k \geq 0$. We remove these known gaps in our new variables by setting $y_1 := x_1$, $y_{k+1} := x_{k+1}$ and $y_i := x_i - 1$ for $2 \leq i \leq k$, which gives

$$\begin{aligned} y_1 + y_2 + \cdots + y_k + y_{k+1} &= x_1 + (x_2 - 1) + \cdots + (x_k - 1) + x_{k+1} \\ &= n - k - (k-1). \end{aligned} \tag{2.7}$$

Finding the number of non-negative integral solutions to this Diophantine equation has many names (the Stars and Bars Problem, Waring's Problem, the Cookie Problem). As the number of solutions to $z_1 + \cdots + z_P = C$ is $\binom{C+P-1}{P-1}$ (see for example [21, 24], or [20] for a proof and an application of this identity in Bayesian analysis), the number of solutions to (2.7) is given by the binomial coefficient

$$\binom{n-k-(k-1)+k}{k} \;=\; \binom{n-(k-1)}{k}.$$

As there are two choices for each a'_{ℓ_i}, we have 2^k legal decompositions whose summands are from the bins $\{\mathcal{B}_{\ell_1}, \mathcal{B}_{\ell_2}, \ldots, \mathcal{B}_{\ell_k}\}$ and thus

$$p_{n,k} \;=\; 2^k \binom{n-(k-1)}{k}.$$

□

3. Gaussian Behavior

Before launching into our proof of Theorem 1.5, we provide some numerical support in Figure 1. We randomly chose 200,000 integers from $[0, 10^{600})$. We observed a mean number of summands of 666.899, which fits beautifully with the predicted value of 666.889; the standard deviation of our sample was 12.154, which is in excellent agreement with the prediction of 12.176.

We split Theorem 1.5 into three parts: a proof of our formula for the mean, a proof of our formula for the variance, and a proof of Gaussian behavior. We isolate the first two as separate propositions; we will prove these after first deriving some useful properties of the generating function of the $p_{n,k}$'s.

Proposition 3.1. *The mean number of summands in the Kentucky legal decompositions for integers in $[0, a_{2n+1})$ is*

$$\mu_n \;=\; \frac{n}{3} + \frac{2}{9} + O\left(\frac{n}{2^n}\right).$$

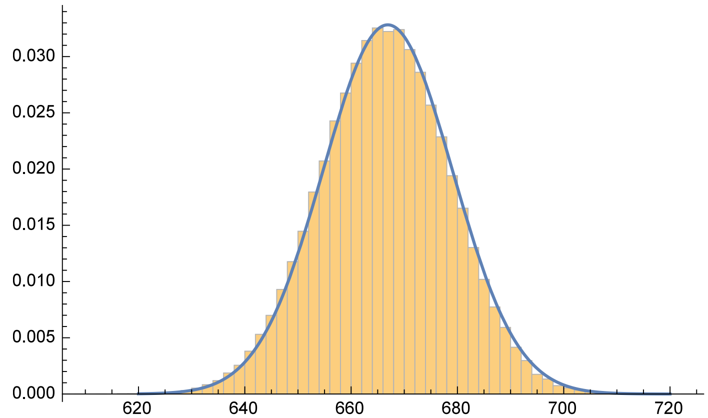

FIGURE 1. The distribution of the number of summands in Kentucky legal decompositions for 200,000 integers from $[0, 10^{600})$.

Proposition 3.2. *The variance σ_n^2 of Y_n (from Theorem 1.5) is*

$$\sigma_n^2 \;=\; \frac{2n}{27} + \frac{8}{81} + O\left(\frac{n^2}{2^n}\right).$$

3.1. **Mean and Variance.** Recall Y_n is the random variable denoting the number of summands in the unique Kentucky decomposition of an integer chosen uniformly from $[0, a_{2n+1})$, and $p_{n,k}$ denotes the number of integers in $[0, a_{2n+1})$ whose legal decomposition contains exactly k summands. The following lemma yields expressions for the mean and variance of Y_n using a generating function for the $p_{n,k}$'s; in fact, it is this connection of derivatives of the generating function to moments that make the generating function approach so appealing. The proof is standard (see for example [9]).

Lemma 3.3. [9, Propositions 4.7, 4.8] *Let $F(x,y) := \sum_{n,k\geq 0} p_{n,k}x^n y^k$ be the generating function of $p_{n,k}$, and let $g_n(y) := \sum_{k=0}^{n} p_{n,k}y^k$ be the coefficient of x^n in $F(x,y)$. Then the mean of Y_n is*

$$\mu_n \;=\; \frac{g_n'(1)}{g_n(1)},$$

and the variance of Y_n is

$$\sigma_n^2 \;=\; \frac{\frac{d}{dy}(y g_n'(y))|_{y=1}}{g_n(1)} - \mu_n^2.$$

In our analysis our closed form expression of $p_{n,k}$ as a binomial coefficient is crucial in obtaining simple closed form expressions for the needed quantities. We are able to express these needed quantities in terms of the **Fibonacci polynomials**, which are defined recursively as follows:

$$F_0(x) \;=\; 0,\; F_1(x) \;=\; 1,\; F_2(x) \;=\; x,$$

and for $n \geq 3$

$$F_n(x) \;=\; xF_{n-1}(x) + F_{n-2}(x).$$

For $n \geq 3$, the Fibonacci polynomial[5] $F_n(x)$ is given by

$$F_n(x) \;=\; \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \tag{3.1}$$

and also has the explicit formula

$$F_n(x) \;=\; \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}. \tag{3.2}$$

The derivative of $F_n(x)$ is given by

$$F_n'(x) \;=\; \frac{2nF_{n-1}(x) + (n-1)xF_n(x)}{x^2+4}. \tag{3.3}$$

For a reference on Fibonacci polynomials and the formulas given above (which follow immediately from the definitions and straightforward algebra), see [19].

Proposition 3.4. *For $n \geq 3$*

$$g_n(y) \;=\; (\sqrt{2y})^{n+1} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right). \tag{3.4}$$

Proof. By Proposition 2.4, we have

$$F(x,y) \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{n,k} x^n y^k \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^k \binom{n-k+1}{k} x^n y^k.$$

Thus, using (3.1) we find

$$\begin{aligned} F(x,y) \;&=\; \frac{1}{x^2\sqrt{2y}} \sum_{n=0}^{\infty}\sum_{k=0}^{n+2} \binom{(n+2)-k-1}{k} \left(\frac{1}{\sqrt{2y}}\right)^{(n+2)-2k-1} (x\sqrt{2y})^{n+2} \\ &=\; \frac{1}{x^2\sqrt{2y}} \sum_{n=0}^{\infty} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right)(x\sqrt{2y})^{n+2} \;=\; \sum_{n=0}^{\infty} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right)(\sqrt{2y})^{n+1}x^n, \end{aligned}$$

completing the proof. □

In Appendix B we provide alternate proofs of Proposition 3.1, Proposition 3.2 and Theorem 1.5 using different methods. In doing so, we uncovered another formula for $g_n(y)$, the coefficient for x^n in $F(x,y)$ as given in Lemma 3.3, and this leads to a derivation of a formula for the Fibonacci polynomials.

[5] Note that $F_n(1)$ gives the standard Fibonacci sequence.

Proof of Proposition 3.1. By Lemma 3.3, the mean of Y_n is $g_n'(1)/g_n(1)$. Calculations of derivatives using equations (3.3) and (3.4) give

$$\begin{aligned}
\frac{g_n'(1)}{g_n(1)} &= \frac{(n+1)(\sqrt{2})^{n-1}F_{n+2}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})(\sqrt{2})^{n+1}} - \frac{(\sqrt{2})^{n-2}F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})(\sqrt{2})^{n+1}} \\
&= \frac{n+1}{2} - \frac{1}{(\sqrt{2})^3}\frac{F_{n+2}'\left(\frac{1}{\sqrt{2}}\right)}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}. \\
&= \frac{n+1}{2} - \frac{2(n+2)F_{n+1}\left(\frac{1}{\sqrt{2}}\right) + \frac{n+1}{\sqrt{2}}F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}{9\sqrt{2}F_{n+2}\left(\frac{1}{\sqrt{2}}\right)} \\
&= \frac{4}{9}(n+1) - \frac{\sqrt{2}}{9}(n+2)\frac{F_{n+1}\left(\frac{1}{\sqrt{2}}\right)}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)} \\
&= \frac{4}{9}(n+1) - \frac{\sqrt{2}}{9}(n+2)\left(\frac{1}{\sqrt{2}} + O(2^{-n})\right) \;=\; \frac{n}{3} + \frac{2}{9} + O(n2^{-n}),
\end{aligned}$$

where in the next-to-last step we use (3.2) to approximate $F_{n+1}(1/\sqrt{2})/F_{n+2}(1/\sqrt{2})$. □

Proof of Proposition 3.2. By Lemma 3.3,

$$\sigma_n^2 \;=\; \frac{g_n''(1)}{g_n(1)} + \frac{g_n'(1)}{g_n(1)} - \mu_n^2 \;=\; \frac{g_n''(1)}{g_n(1)} + \mu_n(1-\mu_n).$$

Now,

$$\frac{g_n''(1)}{g_n(1)} \;=\; \frac{(-2n+1)}{4\sqrt{2}}\frac{F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{(n^2-1)}{4} + \frac{1}{8}\frac{F_{n+2}''(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})}.$$

Applying the derivative formula in (3.3) and using (3.2), we find

$$\begin{aligned}
\frac{F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} &= \frac{4(n+2)}{9}\frac{F_{n+1}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{\sqrt{2}(n+1)}{9} \\
&= \frac{4(n+2)}{9}\left[\frac{1}{\sqrt{2}} + O(2^{-n})\right] + \frac{\sqrt{2}(n+1)}{9}
\end{aligned}$$

and

$$\begin{aligned}
\frac{F_{n+2}''(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} &= \frac{16(n^2+3n+2)}{81}\frac{F_n(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{4\sqrt{2}(2n^2+3n-2)}{81}\frac{F_{n+1}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{2(n^2+9n+8)}{81} \\
&= \frac{16(n^2+3n+2)}{81}\left[\frac{1}{2} + O(2^{-n})\right] + \frac{4\sqrt{2}(2n^2+3n-2)}{81}\left[\frac{1}{\sqrt{2}} + O(2^{-n})\right] \\
&\quad + \frac{2(n^2+9n+8)}{81}.
\end{aligned}$$

Thus

$$\sigma_n^2 = \frac{(-2n+1)}{4\sqrt{2}}\left[\frac{\sqrt{2}}{9}(3n+5)+O(n2^{-n})\right]+\frac{(n^2-1)}{4}+\frac{1}{8}\left[\frac{2n^2}{9}+\frac{2n}{3}+\frac{8}{27}+O(n^2 2^{-n})\right]$$
$$+\left[\frac{n}{3}+\frac{2}{9}+O\left(\frac{n}{2^n}\right)\right]\left[1-\frac{n}{3}-\frac{2}{9}+O\left(\frac{n}{2^n}\right)\right] = \frac{2n}{27}+\frac{8}{81}+O\left(\frac{n^2}{2^n}\right),$$

completing the proof. □

3.2. **Gaussian Behavior.**

Proof of Theorem 1.5. We prove that Y_n' converges in distribution to the standard normal distribution as $n \to \infty$ by showing that the moment generating function of Y_n' converges to that of the standard normal (which is $e^{t^2/2}$). Following the same argument as in [9, Lemma 4.9], the moment generating function $M_{Y_n'}(t)$ of Y_n' is

$$M_{Y_n'}(t) = \frac{g_n(e^{t/\sigma_n})e^{-t\mu_n/\sigma_n}}{g_n(1)}.$$

Thus we have

$$M_{Y_n'}(t) = \frac{F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right)e^{(\frac{n+1}{2}-\mu_n)t/\sigma_n}}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)},$$

and

$$\log(M_{Y_n'}(t)) = \log F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right)+\frac{t}{\sigma_n}\left(\frac{n+1}{2}-\mu_n\right)-\log F_{n+2}\left(\frac{1}{\sqrt{2}}\right).$$

From (3.2),

$$F_{n+2}(x) = \frac{(x+\sqrt{x^2+4})^{n+2}}{2^{n+2}\sqrt{x^2+4}}\left[1-\left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^{n+2}\right].$$

Thus

$$\begin{aligned}\log F_{n+2}(x) &= (n+2)\log(x+\sqrt{x^2+4})-(n+2)\log 2\\ &\quad -\frac{1}{2}\log(x^2+4)+\log(1-r(x)^{n+2})\\ &= (n+2)\log x+(n+2)\log\left(1+\sqrt{1+\frac{4}{x^2}}\right)-(n+2)\log 2\\ &\quad -\frac{1}{2}\log(x^2+4)\ +\ O(r(x)^n),\end{aligned}$$

where for all x

$$r(x) = \left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right) \in (0,1].$$

Thus

$$\log F_{n+2}\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{2}(n+3)\log 2-\log 3+O(2^{-n})$$

and

$$\begin{aligned}\log F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right) &= -\frac{(n+2)}{2}\log 2 - \frac{(n+2)}{2\sigma_n}t - (n+2)\log 2\\ &+ (n+2)\alpha_n(t) - \frac{1}{2}\beta_n(t) + O(r^n),\end{aligned}$$

where

$$\alpha_n(t) = \log(1+\sqrt{1+8e^{t/\sigma_n}}), \quad \beta_n(t) = \log\left(\frac{1}{2}e^{-t/\sigma_n}+4\right),$$

and

$$r = r\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right) < 1.$$

The Taylor series expansions for $\alpha_n(t)$ and $\beta_n(t)$ about $t=0$ are given by

$$\alpha_n(t) = \log 4 + \frac{1}{3\sigma_n}t + \frac{1}{27\sigma_n^2}t^2 + O(n^{-3/2})$$

and

$$\beta_n(t) = \log\left(\frac{9}{2}\right) - \frac{1}{9\sigma_n}t + \frac{4}{81\sigma_n^2}t^2 + O(n^{-3/2}).$$

Going back to $\log(M_{Y_n'}(t))$ we now have

$$\begin{aligned}\log(M_{Y_n'}(t)) &= -\frac{3}{2}(n+2)\log 2 - \frac{(n+2)}{2\sigma_n}t + (n+2)\left[2\log 2 + \frac{1}{3\sigma_n}t + \frac{1}{27\sigma_n^2}t^2 + O(n^{-3/2})\right]\\ &\quad -\frac{1}{2}\left[2\log 3 - \log 2 + O(n^{-1/2})\right] + \frac{(n+1-2\mu_n)}{2\sigma_n}t - \frac{1}{2}(n+3)\log 2 + \log 3\\ &\quad +O(2^{-n}) + O(r^n)\\ &= -\frac{(2\mu_n+1)}{2\sigma_n}t + \frac{(n+2)}{3\sigma_n}t + \frac{(n+2)}{27\sigma_n^2}t^2 + O(n^{-1/2}) + O(2^{-n}) + O(r^n).\end{aligned}$$

Since $\mu_n \sim \frac{n}{3}$ and $\sigma_n^2 \sim \frac{2n}{27}$, it follows that $\log(M_{Y_n'}(t)) \to \frac{1}{2}t^2$ as $n \to \infty$. As this is the moment generating function of the standard normal, our proof is completed. □

4. Average Gap Distribution

In this section we prove our results about the behavior of gaps between summands in Kentucky decompositions. The advantage of studying the average gap distribution is that, following the methods of [2, 5], we reduce the problem to a combinatorial one involving how many $m \in [0, a_{2n+1})$ have a gap of length g starting at a given index i. We then write the gap probability as a double sum over integers m and starting indices i, interchange the order of summation, and invoke our combinatorial results.

While the calculations are straightforward once we adopt this perspective, they are long. Additionally, it helps to break the analysis into different cases depending on the parity of i and g, which we do first below and then use those results to determine the probabilities.

Proof of Theorem 1.6. Let $I_n := [0, a_{2n+1})$ and let $m \in I_n$ with the legal decomposition

$$m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k},$$

with $\ell_1 < \ell_2 < \cdots < \ell_k$. For $1 \le i, g \le n$ we define $X_{i,g}(m)$ as an indicator function which denotes whether the decomposition of m has a gap of length g beginning at i. Formally,

$$X_{i,g}(m) \;=\; \begin{cases} 1 & \text{if } \exists\, j,\ 1 \le j \le k \text{ with } i = \ell_j \text{ and } i+g = \ell_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Notice when $X_{i,g}(m) = 1$, this implies that there exists a gap between a_i and a_{i+g}. Namely m does not contain $a_{i+1}, \ldots, a_{i+g-1}$ as summands in its legal decomposition.

As the definition of the Kentucky sequence implies $P(g) = 0$ for $0 \le g \le 2$, we assume below that $g \ge 3$. Hence if a_j is a summand in the legal decomposition of m and $a_j < a_i$, then the admissible j are at most $i-4$ if and only if i is even, whereas the admissible j are at most $i-3$ if and only if i is odd. We are interested in computing the fraction of gaps (arising from the decompositions of all $m \in I_n$) of length g. This probability is given by

$$P_n(g) \;=\; c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-g} X_{i,g}(m),$$

where

$$c_n \;=\; \frac{1}{(\mu_n - 1)a_{2n+1}}. \tag{4.1}$$

To compute the above-mentioned probability we argue based on the parity of i. We find the contribution of gaps of length g from even i and odd i separately and then add these two. The case when $g = 3$ is a little simpler, as only even i contribute. If i were odd and $g = 3$ we would violate the notion of a Kentucky legal decomposition.

Part 1 of the Proof, Gap Preliminaries:

Case 1, i is even: Suppose that i is even. This means that a_i is the largest entry in its bin. Thus the largest possible summand less than a_i would be a_{i-4}. First we need to know the number of legal decompositions that only contain summands from $\{a_1, \ldots, a_{i-4}\}$, but this equals the number of integers that lie in $[0, a_{2(\frac{i-4}{2})+1}) \;= [0, a_{i-3})$. By (2.1), this is given by

$$a_{2(\frac{i-4}{2})+1} = a_{i-3} \;=\; \frac{1}{3}(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}).$$

Next we must consider the possible summands between a_{i+g} and a_{2n+1}. There are two cases to consider depending on the parity of $i+g$.

Subcase (i), g is even: Notice that in this case $i+g$ is even and if a_j is a summand in the legal decomposition of m with $a_{i+g} < a_j$, then $j \ge i+g+3$. In this case the number of legal decompositions only containing summands from the set $\{a_{i+g+3}, a_{i+g+4}, \ldots, a_{2n}\}$ is the same as the number of integers that lie in $[0, a_{(2n-(i+g+2))+1})$, which equals

$$a_{(2n-(i+g+2))+1} \;=\; a_{2\left(\frac{2n-(i+g+2)}{2}+1\right)-1} \;=\; \frac{1}{3}\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

So for fixed i and g both even, the number of $m \in I_n$ that have a gap of length g beginning at i is

$$\frac{1}{9}\left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Hence in this case we have that

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i,g \text{ even}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i,g \text{ even}}}^{2n-g} \left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Subcase (ii), g is odd: In the case when i is even and g is odd, any legal decomposition of an integer $m \in I_n$ with a gap from i to $i+g$ that contains summands $a_j > a_{i+g}$ must have $j \geq i+g+4$. The number of legal decompositions achievable only with summands in the set $\{a_{i+g+4}, a_{i+g+5}, \ldots, a_{2n}\}$ is the same as the number of integers in the interval $[0, a_{2n-(i+g+2)})$, which is given by

$$a_{2n-(i+g+2)} \;=\; a_{2\left(\frac{2n-(i+g+1)}{2}\right)-1} \;=\; \frac{1}{3}\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Hence when i is even and g is odd we have that

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}, g \text{ odd}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}, g \text{ odd}}}^{2n-g} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}})\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Subcase (iii), $g = 3$: As remarked above, there are no gaps of length 3 when i is odd, and thus the contribution from i even is the entire answer and we can immediately find that

$$\begin{aligned}
P_n(3) \;&=\; c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} X_{i,3}(m) \\
&=\; \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} \left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+4)}{2}+1} + (-1)^{\frac{2n-(i+4)}{2}}\right) \\
&=\; \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} \left(2^{n-1} + 2^{\frac{i}{2}}(-1)^{\frac{2n-(i+4)}{2}} + 2^{\frac{2n-(i+4)}{2}+1}(-1)^{\frac{i-2}{2}} + (-1)^{n-3}\right).
\end{aligned}$$

As the largest term in the above sum is 2^{n-1}, we have

$$P_n(3) = \frac{c_n}{9}\left[(n-1)2^{n-1} + O(2^n)\right].$$

Since $\mu_n \sim \frac{n}{3}$ and $a_{2n+1} \sim \frac{1}{3}(4)(2^n)$, using (4.1) we find that up to lower order terms which vanish as $n \to \infty$ we have

$$c_n \;\sim\; \frac{9}{n2^{n+2}}. \tag{4.2}$$

Therefore

$$P_n(3) \;\sim\; \frac{1}{n2^{n+2}}\left[(n-1)2^{n-1} + O(2^n)\right] \;=\; \left(\frac{1}{8}\right)\left(\frac{n-1}{n}\right) + O\left(\frac{1}{n}\right).$$

Now as n goes to infinity we see that $P(3) = 1/8$.

Case 2, i is odd: Suppose now that i is odd. The largest possible summand less than a_i in a legal decomposition is a_{i-3}. As before we now need to know the number of integers that lie in $[0, a_{2(\frac{i-3}{2})+1})$, but this equals

$$a_{2(\frac{i-3}{2})+1} \;=\; a_{2(\frac{i-1}{2})-1} \;=\; \frac{1}{3}\left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right).$$

We now need to consider the parity of $i+g$.

Subcase (i), g is odd: When i and g are odd, we know $i+g$ is even and therefore the first possible summand greater than a_{i+g} is a_{i+g+3}. Like before, the number of legal decompositions using summands from the set $\{a_{i+g+3}, a_{i+g+4}, \dots, a_{2n}\}$ is the same as the number of m with legal decompositions using summands from the set $\{a_1, a_2, \dots, a_{2n-(i+g+2)}\}$, which is $\frac{1}{3}\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right)$. This leads to

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ odd}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ odd}}}^{2n-g} \left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Subcase (ii), g is even: Following the same line of argument we see that if i is odd and g is even, then

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ even}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ even}}}^{2n-g} \left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right)\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Using these results, we can combine the various cases to determine the gap probabilities for different g.

Part 2 of the Proof, Gap Probabilities:

Case 1, g is even: As g is even, we have $g = 2j$ for some positive integer j. Therefore

$$\begin{aligned}
P_n(2j) &= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-2j} X_{i,2j}(m) \\
&= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} X_{i,2j}(m) + c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} X_{i,2j}(m) \\
&= c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}})(2^{\frac{2n-(i+2j)}{2}+1} + (-1)^{\frac{2n-(i+2j)}{2}}) \right] \\
&\quad + c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} (2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}})(2^{\frac{2n-(i+2j+1)}{2}+1} + (-1)^{\frac{2n-(i+2j+1)}{2}}) \right] \\
&= \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} (2^{n-j+1} + 2^{\frac{i}{2}}(-1)^{\frac{2n-(i+2j)}{2}} + 2^{\frac{2n-(i+2j)}{2}+1}(-1)^{\frac{i-2}{2}} + (-1)^{n-j-1}) \\
&\quad + \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} (2^{n-j+1} + 2^{\frac{i-1}{2}+1}(-1)^{\frac{2n-(i+2j+1)}{2}} + 2^{\frac{2n-(i+2j+1)}{2}+1}(-1)^{\frac{i-1}{2}} + (-1)^{n-j-1}).
\end{aligned}$$

Notice that the largest terms in the above sums/expressions are given by 2^{n-j+1} and 2^{n-j+1}, the sum of which gives $4(n-j)2^{n-j}$. The rest of the terms are of lower order and are dominated as $n \to \infty$. Using (4.2) for c_n we find

$$P_n(2j) \sim \frac{c_n}{9} 4(n-j)2^{n-j} \sim \left(\frac{1}{n2^{n+2}} \right) 4(n-j)2^{n-j} = \frac{n-j}{n2^j},$$

and thus as n goes to infinity we see that $P(2j) = 1/2^j$.

Case 2, g is odd: As g is odd we may write $g = 2j + 1$. Thus

$$\begin{aligned}
P_n(2j+1) &= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-2j-1} X_{i,2j+1}(m) \\
&= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} X_{i,2j+1}(m) + c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} X_{i,2j+1}(m) \\
&= c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}) \left(2^{\frac{2n-(i+2j+2)}{2}+1} + (-1)^{\frac{2n-(i+2j+2)}{2}}\right) \right] \\
&\quad + c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} (2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}})(2^{\frac{2n-(i+2j+1)}{2}+1} + (-1)^{\frac{2n-(i+2j+1)}{2}}) \right] \\
&= \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} \left(2^{n-j} + 2^{\frac{i}{2}}(-1)^{\frac{2n-(i+2j+2)}{2}} + 2^{\frac{2n-(i+2j+2)}{2}+1}(-1)^{\frac{i-2}{2}} + (-1)^{n-j-2}\right) \\
&\quad + \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} \left(2^{n-j+1} + 2^{\frac{i-1}{2}+1}(-1)^{\frac{2n-(i+2j+1)}{2}} + 2^{\frac{2n-(i+2j+1)}{2}+1}(-1)^{\frac{i-1}{2}} + (-1)^{n-j-1}\right).
\end{aligned}$$

Notice that the largest terms in the above sums/expressions are given by 2^{n-j} and 2^{n-j+1}, the sum of which gives $3(n-j)2^{n-j}$. The rest of the terms are of lower order and are dominated as $n \to \infty$. Using (4.2) for c_n we find

$$P_n(2j+1) \sim \frac{c_n}{9} 3(n-j)2^{n-j} \sim \left(\frac{1}{n2^{n+2}}\right) 3(n-j)2^{n-j} = \left(\frac{3}{4}\right)\left(\frac{n-j}{n2^j}\right),$$

and thus as n goes to infinity we see that $P(2j+1) = \frac{3}{4}\left(1/2^j\right)$. □

5. Conclusion and Future Work

Our results generalize Zeckendorf's theorem to an interesting new class of recurrence relations, specifically to a case where the first coefficient is zero. While we still have uniqueness of decompositions in the Kentucky sequence, that is not always the case for this class of recurrences. In a future work [6] we study another example with first coefficient zero, the recurrence $a_{n+1} = a_{n-1} + a_{n-2}$. This leads to what we call the Fibonacci Quilt, and there uniqueness of decomposition fails. The non-uniqueness gives rise to new interesting discussions, for example the handling of the question of Gaussian behavior for the distribution of the number of summands given that we now have multiple decompositions for most integers; we address these issues in [6].

Additionally, the Kentucky sequence is but one of infinitely many (s, b)-Generacci sequences; in a future work [7] we hope to give a detailed study of these sequences and to extend the results of this paper to arbitrary (s, b). The difficulty is that many of the arguments in the paper here crucially use explicit formulas available for quantities associated to the Kentucky sequence, which are not known for general sequences. This difficulty mirrors the difference between [18] (which used binomial coefficient expressions from the Zeckendorf decompositions) and [22] (the general case required many technical arguments).

Definition 5.1. *Let an increasing sequence of positive integers* $\{a_i\}_{i=1}^{\infty}$ *and a family of subsequences* $\mathcal{B}_n = \{a_{b(n-1)+1}, \ldots, a_{bn}\}$ *be given. (We call these subsequences* bins*.) We declare a decomposition of an integer* $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ *where* $a_{\ell_i} < a_{\ell_{i+1}}$ *to be a* (s,b)-Generacci decomposition *provided* $\{a_{\ell_i}, a_{\ell_{i+1}}\} \not\subset \mathcal{B}_{j-s} \cup \mathcal{B}_{j-s+1} \cup \cdots \cup \mathcal{B}_j$ *for all* i, j*. (We say* $\mathcal{B}_j = \emptyset$ *for* $j \le 0$*.)*

This says that for all $a_{\ell_i} \in \mathcal{B}_j$, no other $a_{\ell_{i'}}$ is also in the jth bin nor in any of the adjacent s bins preceding $\mathcal{B}_j$ nor the s bins succeeding $\mathcal{B}_j$.

Definition 5.2. *An increasing sequence of positive integers* $\{a_i\}_{i=1}^{\infty}$ *is called an* (s,b)-Generacci sequence *if every* a_i *for* $i \ge 1$ *is the smallest positive integer that does not have a* (s,b)*-Generacci legal decomposition using the elements* $\{a_1, \ldots, a_{i-1}\}$.

Note that we still have uniqueness of decompositions as in Theorem 1.4; this follows from Theorem 1.3 of [9]. Numerical simulations suggest that the number of summands in the unique (s,b)-Generacci decomposition of a positive integer exhibits Gaussian behavior. The Fibonacci polynomial approach in Section 3 extends nicely for general b, thus proving Gaussianity for all $(1,b)$-Generacci sequences. The technique however fails to generalize for $s > 1$. We are investigating methods to attack the general case.

Appendix A. Unique Decompositions

Proof of Theorem 1.4. Our proof is constructive. We build our sequence by only adjoining terms that ensure that we can *uniquely* decompose a number while never using more than one summand from the same bin or two summands from adjacent bins. The sequence begins:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \ldots.$$

Note we would not adjoin 9 because then 9 would legally decompose two ways, as $9 = 9$ and as $9 = 8 + 1$. The next number in the sequence must be the smallest integer that cannot be decomposed legally using the current terms.

We proceed with proof by induction. The base case follows from a direct calculation. Notice that if $i \le 5$ then $i = a_i$. Also $6 = a_5 + a_1$.

The sequence continues:

$$\ldots,\ \underbrace{a_{2n-5},\ a_{2n-4}}_{\mathcal{B}_{n-2}},\ \underbrace{a_{2n-3},\ a_{2n-2}}_{\mathcal{B}_{n-1}},\ \underbrace{a_{2n-1},\ a_{2n}}_{\mathcal{B}_{n}},\ \underbrace{a_{2n+1},\ a_{2n+2}}_{\mathcal{B}_{n+1}},\ \ldots$$

By induction we assume that there exists a unique decomposition for all integers $m \le a_{2n} + w$, where w is the maximum integer that legally can be decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n-4}\}$. By construction we know that $w = a_{2n-3} - 1$, as this was the reason we adjoined a_{2n-3} to the sequence.

Now let y be the maximum integer that can be legally decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n}\}$. By construction we have

$$y \ = \ a_{2n} + w \ = \ a_{2n} + a_{2n-3} - 1.$$

Similarly, let x be the maximum integer that legally can be decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n-2}\}$. Note $x = a_{2n-1} - 1$ as this is why we include a_{2n-1} in the sequence.

Claim: $a_{2n+1} = y + 1$ and this decomposition is unique.

By induction we know that y was the largest value that we could legally make using only terms in $\{a_1, a_2, \dots, a_{2n}\}$. Hence we choose $y+1$ as a_{2n+1} and $y+1$ has a unique decomposition.

Claim: All $N \in [y+1, y+1+x] = [a_{2n+1}, a_{2n+1}+x]$ have a unique decomposition.

We can legally and uniquely decompose all of $1, 2, 3, \dots, x$ using elements in the set $\{a_1, a_2, \dots, a_{2n-2}\}$. Adding a_{2n+1} to the decomposition is still legal since a_{2n+1} is not a member of any bins adjacent to $\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{n-1}\}$. The uniqueness follows from the fact that if we do not include a_{2n+1} as a summand, then the decomposition does not yield a number big enough to exceed $y+1$.

Claim: $a_{2n+2} = y + 1 + x + 1 = a_{2n+1} + x + 1$ and this decomposition is unique.

By construction the largest integer that legally can be decomposed using terms $\{a_1, a_2, \dots, a_{2n+1}\}$ is $y + 1 + x$.

Claim: All $N \in [a_{2n+2}, a_{2n+2} + x]$ have a unique decomposition.

First note that the decomposition exists as we can legally and uniquely construct $a_{2n+2} + v$, where $0 \le v \le x$. For uniqueness, we note that if we do not use a_{2n+2}, then the summation would be too small.

Claim: $a_{2n+2} + x$ is the largest integer that legally can be decomposed using terms $\{a_1, a_2, \dots, a_{2n+2}\}$.

This follows from construction. □

Appendix B. Generating Function Proofs

In §3 we proved that the distribution of the number of summands in a Kentucky decomposition exhibits Gaussian behavior by using properties of Fibonacci polynomials. This approach was possible because we had an explicit, tractable form for the $p_{n,k}$'s (Proposition 2.4) that coincided with the explicit sum formulas associated with the Fibonacci polynomials. Below we present a second proof of Gaussian behavior using a more general approach, which might be more useful in addressing the behavior of the number of summands when dealing with general (s,b)-Generacci sequences.

As in the first proof, we are interested in $g_n(y)$, the coefficient of the x^n term in $F(x,y)$.

Lemma B.1. *We have*

$$\begin{aligned} g_n(y) \quad &= \quad \frac{1}{2^{n+1}\sqrt{1+8y}} \left[4y\left(1+\sqrt{1+8y}\right)^n - 4y\left(1-\sqrt{1+8y}\right)^n \right. \\ &\quad \left. + \ \left(1+\sqrt{1+8y}\right)^{n+1} - \left(1-\sqrt{1+8y}\right)^{n+1} \right]. \end{aligned} \tag{B.1}$$

Proof. For brevity set $x_1 = x_1(y)$ and $x_2 = x_2(y)$ for the roots of x in $x^2 + \frac{1}{2y}x - \frac{1}{2y}$. In particular, we find

$$x_1 \ = \ -\frac{1}{4y}\left(1+\sqrt{1+8y}\right) \qquad x_2 \ = \ -\frac{1}{4y}\left(1-\sqrt{1+8y}\right). \tag{B.2}$$

Since x_1 and x_2 are unequal for all $y > 0$, we can decompose $F(x,y)$ using partial fractions:

$$F(x,y) \ = \ \frac{1+2xy}{-2y(x-x_1)(x-x_2)} \ = \ \frac{1+2xy}{-2y}\frac{1}{x_1-x_2}\left[\frac{1}{x-x_1} - \frac{1}{x-x_2}\right].$$

Using the geometric series formula, after some algebra we obtain

$$F(x,y) \;=\; \frac{1+2xy}{-2y}\frac{1}{x_1-x_2}\sum_{i\geq 0}\left[\frac{1}{x_1}\left(\frac{x}{x_1}\right)^i-\frac{1}{x_2}\left(\frac{x}{x_2}\right)^i\right].$$

From here we find that that the coefficient of x^n is

$$g_n(y) \;=\; \frac{1}{-2y(x_1-x_2)}\left[\frac{1}{x_1^{n+1}}-\frac{1}{x_2^{n+1}}+\frac{2y}{x_1^n}-\frac{2y}{x_2^n}\right].$$

Substituting the functions from (B.2) and simplifying we obtain the desired result. $\square$

As we mentioned in §3.1, we have the following corollary.

Corollary B.2. *Let $F_n(x)$ be a Fibonacci polynomial. Then*

$$F_n(x) \;=\; \frac{(x+\sqrt{x^2+4})^n-(x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}.$$

Proof. Set the right hand sides of equations (3.4) and (B.1) equal and let $x = 1/\sqrt{2y}$. $\square$

Proof of Proposition 3.1. Straightforward, but somewhat tedious, calculations give

$$\begin{aligned} g_n(1) &= \frac{1}{3}\left((-1)^{n+1}+2^{n+2}\right)\\ g_n'(1) &= \frac{n}{9}\left(2^{n+2}+2(-1)^{n+1}\right)+\frac{2}{27}\left(2^{n+2}\right)+o(1). \end{aligned}$$

Dividing these two quantities and using Lemma 3.3 gives the desired result. $\square$

Proof of Proposition 3.2. Another straightforward (and again somewhat tedious) calculation yields

$$\begin{aligned} \sigma_n^2 &= \frac{2^{2n+5}(4+3n)-2(8+3n)-2^{n+2}(-1)^n(28+36n+9n^2)}{81(2^{n+2}-(-1)^n)^2}\\ &= \frac{n\big[(6)2^{2n+4}-18(-1)^n2^{n+3}-6\big]+\big[(8)2^{2n+4}-14(-1)^n2^{n+3}-16\big]-4.5(-1)^nn^22^{n+3}}{81\big[2^{2n+4}-(-1)^n2^{n+3}+1\big]}. \end{aligned}$$

$\square$

Proof of Theorem 1.5. As in our earlier proof, we show that the moment generating function of Y_n' converges to that of the standard normal. Following the same argument as in [9, Lemma 4.9], the moment generating function $M_{Y_n'}(t)$ of Y_n' is

$$M_{Y_n'}(t) \;=\; \frac{g_n(e^{t/\sigma_n})e^{-t\mu_n/\sigma_n}}{g_n(1)}.$$

Taking logarithms yields

$$\log M_{Y_n'}(t) \;=\; \log[g_n(e^{t/\sigma_n})]-\log[g_n(1)]-\frac{t\mu_n}{\sigma_n}. \tag{B.3}$$

We tackle the right hand side in pieces.

Let $r_n = t/\sigma_n$. Since $\sigma_n^2 = \frac{2n}{27}+\frac{8}{81}+O\left(\frac{n^2}{2^n}\right)$, as n goes to infinity r_n goes to 0. This allows us to use Taylor series expansions.

First we rewrite $g_n(e^{r_n})$

$$\begin{aligned} g_n(e^{r_n}) =&\frac{1}{\sqrt{1+8e^{r_n}}}\left[\frac{(1+\sqrt{1+8e^{r_n}})^n(4e^{r_n}+1+\sqrt{1+8e^{r_n}})}{2^{n+1}}\right.\\ &\left.-\frac{4e^{r_n}(1-\sqrt{1+8e^{r_n}})^n}{2^{n+1}}-\frac{(1-\sqrt{1+8e^{r_n}})^{n+1}}{2^{n+1}}\right]. \end{aligned}$$

Using Taylor series expansions of the exponential and square root functions we obtain

$$e^{r_n}=1+o(1) \quad \text{and} \quad \frac{1-\sqrt{1+8e^{r_n}}}{2}=-1+o(1).$$

Thus

$$\begin{aligned} \frac{4e^{r_n}(1-\sqrt{1+8e^{r_n}})^n}{2^{n+1}}+\frac{(1-\sqrt{1+8e^{r_n}})^{n+1}}{2^{n+1}} &= 2(-1)^n+o(1)-(-1)^n+o(1)\\ &= (-1)^n+o(1). \end{aligned}$$

Hence

$$g_n(e^{r_n})=\frac{1}{\sqrt{1+8e^{r_n}}}\left[\frac{(1+\sqrt{1+8e^{r_n}})^n(4e^{r_n}+1+\sqrt{1+8e^{r_n}})}{2^{n+1}}-(-1)^n+o(1)\right].$$

So

$$\begin{aligned} \log(g_n(e^{r_n})) &= -\tfrac{1}{2}\log(1+8e^{r_n})+n\log(1+\sqrt{1+8e^{r_n}})\\ &\quad +\log(4e^{r_n}+1+\sqrt{1+8e^{r_n}})-(n+1)\log 2+o(1). \end{aligned}$$

Continuing to use Taylor series expansions

$$\begin{aligned} \log(g_n(e^{r_n})) =& -\tfrac{1}{2}\left[\log 9+\frac{8}{9}r_n+\frac{4}{81}r_n^2\right]+n\left[\log 4+\frac{1}{3}r_n+\frac{1}{27}r_n^2\right]\\ &+\left[\log 8+\frac{2}{3}r_n+\frac{2}{27}r_n^2\right]+O(r_n^3)-(n+1)\log 2+o(1). \end{aligned} \tag{B.4}$$

Finally, recall $g_n(1)=\frac{1}{3}[(-1)^{n+1}+2^{n+2}]$ so

$$\log[g_n(1)] = -\log 3+(n+2)\log 2+o(1). \tag{B.5}$$

To finish we plug values into (B.3). In particular, plug in $\log(g_n(e^{r_n}))$ from (B.4), $\log[g_n(1)]$ from (B.5), μ_n from Proposition 3.1, σ_n from Proposition 3.2, and $r_n=t/\sigma_n$. This gives

$$\log M_{Y_n'}(t)=\frac{t^2}{2}+o(1).$$

Thus, $M_{Y_n'}(t)$ converges to the moment generating function of the standard normal distribution. Which according to probability theory, implies that the distribution of Y_n' converges to the standard normal distribution. □

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