HIGHER-ORDER IDENTITIES FOR FIBONACCI NUMBERS

TAKAO KOMATSU, ZUZANA MASÁKOVÁ, AND EDITA PELANTOVÁ

Abstract. Let F_n be the n-th Fibonacci number. In this paper, we give some explicit expressions of $\sum_{l=0}^{2r-3}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$ as well as $\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$.

1. Introduction

It is known that the generating function $f(x)$ of Fibonacci numbers F_n is given by

$$f(x):=\frac{x}{1-x-x^2}=\sum_{n=0}^\infty F_n x^n\,.$$

Then $f(x)$ satisfies the relation

$$f(x)^2=\frac{x^2}{1+x^2}f'(x) \tag{1.1}$$

or

$$(1+x^2)f(x)^2=x^2 f'(x)\,. \tag{1.2}$$

The left-hand side of (1.2) is

$$\begin{aligned}
&(1+x^2)\left(\sum_{n=0}^\infty F_n x^n\right)\left(\sum_{m=0}^\infty F_m x^m\right)\\
&=(1+x^2)\sum_{n=0}^\infty\sum_{j=0}^n F_j F_{n-j}x^n\\
&=\sum_{n=0}^\infty\sum_{j=0}^n F_j F_{n-j}x^n+\sum_{n=2}^\infty\sum_{j=0}^{n-2} F_j F_{n-j-2}x^n\,.
\end{aligned}$$

The right-hand side of (1.2) is

$$x^2\left(\sum_{n=1}^\infty nF_n x^{n-1}\right)=\sum_{n=1}^\infty (n-1)F_{n-1}x^n\,.$$

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Comparing the coefficients of both sides, we get

$$\begin{aligned}(n-1)F_{n-1} &= \sum_{j=0}^{n} F_j F_{n-j} + \sum_{j=0}^{n-2} F_j F_{n-j-2} \\ &= \sum_{j=1}^{n-1} F_j F_{n-j} + \sum_{j=0}^{n-2} F_j F_{n-j-2} \\ &= \sum_{j=1}^{n-1} (F_j F_{n-j} + F_{j-1} F_{n-j-1}) \,.\end{aligned}$$

Hence, we get the identity which can be identical with $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ (see e.g. [1, Lemma 5].

Theorem 1.1. *For $n \ge 1$, we have*

$$nF_n = \sum_{j=1}^{n} (F_j F_{n-j+1} + F_{j-1} F_{n-j}) \,.$$

Differentiating both sides of (1.1) by x and dividing them by 2, we obtain

$$f(x)f'(x) = \frac{x}{(1+x^2)^2} f'(x) + \frac{x^2}{2(1+x^2)} f''(x) \,. \tag{1.3}$$

By (1.1) and (1.3), we get

$$\begin{aligned} f(x)^3 &= \frac{x^2}{1+x^2} f(x) f'(x) \\ &= \frac{x^3}{(1+x^2)^3} f'(x) + \frac{x^4}{2(1+x^2)^2} f''(x) \end{aligned} \tag{1.4}$$

or

$$(1+x^2)^3 f(x)^3 = x^3 f'(x) + \frac{1}{2} x^4 (1+x^2) f''(x) \,. \tag{1.5}$$

The left-hand side of (1.5) is

$$\begin{aligned} &(1+3x^2+3x^4+x^6) \sum_{n=0}^{\infty} \sum_{\substack{j_1+j_2+j_3=n \\ j_1,j_2,j_3 \ge 0}} F_{j_1} F_{j_2} F_{j_3} x^n \\ &= \sum_{l=0}^{3} \sum_{n=2l}^{\infty} \binom{l}{3} \sum_{\substack{j_1+j_2+j_3=n-2l \\ j_1,j_2,j_3 \ge 1}} F_{j_1} F_{j_2} F_{j_3} x^n \,. \end{aligned}$$

The right-hand side of (1.5) is

$$\begin{aligned} &x^3 \sum_{n=1}^{\infty} nF_n x^{n-1} + \frac{x^4}{2} \sum_{n=2}^{\infty} n(n-1) F_n x^{n-2} + \frac{x^6}{2} \sum_{n=2}^{\infty} n(n-1) F_n x^{n-2} \\ &= \sum_{n=2}^{\infty} \frac{(n-1)(n-2)}{2} F_{n-2} x^n + \sum_{n=4}^{\infty} \frac{(n-4)(n-5)}{2} F_{n-4} x^n \,. \end{aligned}$$

Comparing the coefficients of both sides, we get the following.

Theorem 1.2. *For $n \geq 6$, we have*

$$\sum_{l=0}^{3} \binom{3}{l} \sum_{\substack{j_1+j_2+j_3=n-2l \\ j_1,j_2,j_3 \geq 1}} F_{j_1} F_{j_2} F_{j_3} = \binom{n-1}{2} F_{n-2} + \binom{n-4}{2} F_{n-4} \,.$$

In this paper, we give some explicit expressions of $\sum_{l=0}^{2r-3} \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r}$ as well as $\sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r}$.

2. Main result

In general, we can state the following.

Theorem 2.1. *Let $r \geq 2$. Then for $n \geq 3r-5$, we have*

$$\sum_{l=0}^{2r-3} \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r} = \sum_{k=1}^{r-1} \frac{n-2k-r+3}{r-1} \binom{n-2k+1}{r-k-1} \binom{n-k-2r+3}{k-1} F_{n-2k-r+3} \,.$$

Lemma 2.2. *For $r \geq 2$, we have*

$$f(x)^r = \frac{x^{2r-2} f^{(r-1)}(x)}{(r-1)!(1+x^2)^{r-1}} + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \binom{r-2}{k-j-1} x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k-1)}(x) \,. \tag{2.1}$$

Proof. The proof is done by induction. It is trivial to see that the identity holds for $r = 2$. Suppose that the identity holds for some r. Differentiating both sides by x, we obtain

$$
\begin{aligned}
&rf(x)^{r-1}f'(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + \frac{(2r-2)x^{2r-3}f^{(r-1)}(x)}{(r-1)!(1+x^2)^{r}}\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j(2r-k-2+2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-3+2j}}{k(r-k-2)!(1+x^2)^{r+k}} f^{(r-k-1)}(x)\\
&\quad - \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j(3k-2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-1+2j}}{k(r-k-2)!(1+x^2)^{r+k}} f^{(r-k-1)}(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + \frac{2x^{2r-3}f^{(r-1)}(x)}{(r-2)!(1+x^2)^{r}}\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1} \frac{\sum_{j=0}^{k-2}(-1)^j(2r-k-1+2j)\binom{k-1}{j}\binom{r-2}{k-j-2}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1} \frac{\sum_{j=1}^{k-1}(-1)^j(3k-2j-1)\binom{k-1}{j-1}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + r\sum_{k=1}^{r-1} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\,.
\end{aligned}
$$

Here, we used the relations

$$\frac{2}{(r-2)!} + \frac{1}{(r-3)!} = \frac{r}{(r-2)!} \quad (k=1)$$

and

$$
\begin{aligned}
&\frac{r-k-1}{k}\binom{k}{j}\binom{r-2}{k-j-1} + \frac{2r-k-1+2j}{k-1}\binom{k-1}{j}\binom{r-2}{k-j-2}\\
&\qquad + \frac{3k-2j-1}{k-1}\binom{k-1}{j-1}\binom{r-2}{k-j-1}\\
&= \frac{r}{k}\binom{k}{j}\binom{r-1}{k-j-1} \quad (k \ge 2)\,.
\end{aligned}
$$

Together with (1.1), we get

$$\begin{aligned} f(x)^{r+1} &= \frac{x^2}{1+x^2} f(x)^{r-1} f'(x) \\ &= \frac{x^2}{1+x^2}\left(\frac{x^{2r-2}f^{(r)}(x)}{r!(1+x^2)^{r-1}} + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\right) \\ &= \frac{x^{2r}f^{(r)}(x)}{r!(1+x^2)^r} + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k+2j}}{k(r-k-1)!(1+x^2)^{r+k}} f^{(r-k)}(x)\,. \end{aligned}$$

□

Proof of Theorem 2.1. By Lemma 2.2 we get

$$\begin{aligned} (1+x^2)^{2r-3} f(x)^r &= (1+x^2)^{r-2}\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!} \\ &\quad + \sum_{k=1}^{r-2}(1+x^2)^{r-k-2}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!} f^{(r-k-1)}(x)\,. \end{aligned} \quad (2.2)$$

Since $F_0 = 0$, the left-hand side of (2.2) is equal to

$$\begin{aligned} &(1+x^2)^{2r-3}\sum_{n=0}^{\infty}\sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\ldots,j_r\ge 0}} F_{j_1}\cdots F_{j_r}x^n \\ &= \sum_{l=0}^{2r-3}\sum_{n=2l}^{\infty}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\ldots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}x^n\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(1+x^2)^{r-2}\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!} \\ &= \sum_{i=0}^{r-2}\binom{r-2}{i}x^{2i}\frac{x^{2r-2}}{(r-1)!}\sum_{n=r-1}^{\infty}\frac{n!}{(n-r+1)!}F_n x^{n-r+1} \\ &= \frac{1}{(r-1)!}\sum_{i=0}^{r-2}\binom{r-2}{i}\sum_{n=2r+2i-2}^{\infty}\frac{(n-r-2i+1)!}{(n-2r-2i+2)!}F_{n-r-2i+1}x^n\,. \end{aligned}$$

For $i = r-2$, we have

$$\begin{aligned} &\frac{1}{(r-1)!}\sum_{n=4r-6}\frac{(n-3r+5)!}{(n-4r+6)!}F_{n-3r+5}x^n \\ &= \sum_{n=3r-5}\frac{n-3r+5}{r-1}\binom{n-3r+4}{r-2}F_{n-3r+5}x^n\,, \end{aligned}$$

which yields the term for $k = r-1$ on the right-hand side of the identity in Theorem 2.1. Notice that

$$\binom{\gamma'}{\gamma} = 0 \quad (\gamma' < \gamma)\,.$$

The second term of the right-hand side of (2.2) is

$$\sum_{k=1}^{r-2}\sum_{i=0}^{r-k-2}\binom{r-k-2}{i}x^{2i}\frac{1}{k(r-k-2)!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}$$
$$\times\sum_{n=r-k-1}^{\infty}\frac{n!}{(n-r+k+1)!}F_n x^{n-r+k+1}$$
$$=\sum_{i=0}^{r-3}\sum_{j=0}^{r-i-3}\sum_{k=j}^{r-i-3}\frac{1}{(k+1)(r-k-3)!(n-2r+k-2i-2j+3)!}\binom{r-k-3}{i}$$
$$\times\binom{k+1}{j}\binom{r-2}{k-j}\sum_{n=2r+2i+2j-k-3}(-1)^j(n-r-2i-2j+1)!F_{n-r-2i-2j+1}x^n$$
$$=\sum_{i=0}^{r-3}\sum_{\kappa=i+1}^{r-2}\sum_{k=\kappa-i-1}^{r-i-3}\frac{1}{(k+1)(r-k-3)!(n-2r+k-2\kappa+5)!}\binom{r-k-3}{i}$$
$$\times\binom{k+1}{\kappa-i-1}\binom{r-2}{k-\kappa+i+1}\sum_{n=2r+2\kappa-k-5}(-1)^{\kappa-i-1}(n-r-2\kappa+3)!F_{n-r-2\kappa+3}x^n\,.$$

Together with the first term of the right-hand side of (2.2) we can prove that

$$\frac{1}{(r-1)!}\binom{r-2}{k-1}\frac{(n-r-2k+3)!}{(n-2r-2k+4)!}$$
$$+\sum_{i=0}^{k-1}\sum_{l=k-i-1}^{r-i-3}\frac{1}{(l+1)(r-l-3)!(n-2r+l-2k+5)!}$$
$$\times\binom{r-l-3}{i}\binom{l+1}{k-i-1}\binom{r-2}{l-k+i+1}(-1)^{k-i-1}(n-r-2k+3)!$$
$$=\frac{n-2k-r+3}{r-1}\binom{n-2k+1}{r-k-1}\binom{n-k-2r+3}{k-1}\,. \tag{2.3}$$

Then the proof is done. □

3. Examples 1

When $r=2$ and $r=3$, Theorem 2.1 is reduced to Theorem 1.1 and Theorem 1.2, respectively. When $r=3,4,5$ in Theorem 2.1, we get the following Corollaries as examples.

Theorem 3.1. *For $n\ge 7$, we have*

$$\sum_{l=0}^{5}\binom{5}{l}\sum_{\substack{j_1+j_2+j_3+j_4=n-2l\\ j_1,j_2,j_3,j_4\ge 1}}F_{j_1}F_{j_2}F_{j_3}F_{j_4}$$
$$=\binom{n-1}{3}F_{n-3}+\frac{(n-3)(n-5)(n-7)}{3}F_{n-5}+\binom{n-7}{3}F_{n-7}\,.$$

Theorem 3.2. *For $n \ge 10$, we have*

$$\sum_{l=0}^{7}\binom{7}{l}\sum_{\substack{j_1+j_2+j_3+j_4+j_5=n-2l\\ j_1,j_2,j_3,j_4,j_5\ge 1}} F_{j_1}F_{j_2}F_{j_3}F_{j_4}F_{j_5}$$
$$=\binom{n-1}{4}F_{n-4}+\frac{(n-3)(n-4)(n-6)(n-9)}{8}F_{n-6}$$
$$+\frac{(n-5)(n-8)(n-10)(n-11)}{8}F_{n-6}+\binom{n-10}{4}F_{n-10}\,.$$

Theorem 3.3. *For $n \ge 13$, we have*

$$\sum_{l=0}^{9}\binom{9}{l}\sum_{\substack{j_1+\cdots+j_6=n-2l\\ j_1,\dots,j_6\ge 1}} F_{j_1}\cdots F_{j_6}$$
$$=\binom{n-1}{5}F_{n-5}+\frac{(n-3)(n-4)(n-5)(n-7)(n-11)}{30}F_{n-7}$$
$$+\frac{(n-5)(n-6)(n-9)(n-12)(n-13)}{20}F_{n-9}$$
$$+\frac{(n-7)(n-11)(n-13)(n-14)(n-15)}{30}F_{n-11}+\binom{n-13}{5}F_{n-13}\,.$$

4. Another result

In this section, we shall give an expression of $\sum_{\substack{j_1+\cdots+j_r\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$.

The left-hand side of (1.1) is

$$\left(\sum_{n=0}^{\infty}F_n x^n\right)\left(\sum_{m=0}^{\infty}F_m x^m\right)=\sum_{n=0}^{\infty}\sum_{j=0}^{n}F_jF_{n-j}x^n\,.$$

The right-hand side of (1.1) is

$$x^2\left(\sum_{j=0}^{\infty}(-1)^j x^{2j}\right)\left(\sum_{m=1}^{\infty}mF_m x^{m-1}\right)=x^2\left(\sum_{j=0}^{\infty}\alpha_j x^j\right)\left(\sum_{m=0}^{\infty}(m+1)F_{m+1}x^m\right)$$
$$=x^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\alpha_{n-m}(m+1)F_{m+1}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n-2}\alpha_{n-m-2}(m+1)F_{m+1}x^n\,,$$

where $\alpha_j=\cos\frac{j\pi}{2}$ $(j\ge 0)$, satisfying $\{\alpha_j\}_{j\ge 0}=1,0,-1,0,1,0,-1,0,1,0,-1,0,\dots$. Comparing the coefficients of both sides, we have the following.

Theorem 4.1. *For $n \ge 2$, we have*

$$\sum_{j=0}^{n} F_j F_{n-j} = \sum_{m=1}^{n-1} m F_m \cos\frac{(n-m-1)\pi}{2}\,. \tag{4.1}$$

In general, we have the following.

Theorem 4.2. *For $n \ge r \ge 2$, we have*

$$\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$$
$$= \frac{C_{r-2}}{(2r-4)!2^{2r-4}} \sum_{m=1}^{n-r+1} \frac{(n+m+r-3)!!(n-m+r-3)!!}{(n+m-r+1)!!(n-m-r+1)!!} m F_m \cos\frac{(n-m-r+1)\pi}{2}\,,$$

where C_n is the n-th Catalan number ([4, A000108]*) given by*

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad (n \ge 0)$$

and $n!! = n(n-2)(n-4)\cdots 1$ if n is odd; $n!! = n(n-2)(n-4)\cdots 2$ if n is even.

Proof. The left-hand side of (2.1) in Lemma 2.2 is equal to

$$\sum_{n=0}^{\infty} \sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r} x^n\,.$$

The first term on the right-hand side of (2.1) in Lemma 2.2 is equal to

$$
\begin{aligned}
&\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!(1+x^2)^{r-1}}\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{i=0}^{\infty}\binom{i+r-2}{r-2}x^{2i}\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}F_{m+r-1}x^m\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{k=0}^{\infty}\frac{1}{(r-2)!2^{r-2}}\frac{(k+2r-4)!!}{k!!}\cos\frac{k\pi}{2}x^k\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}F_{m+r-1}x^m\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(r-2)!2^{r-2}}\frac{(n-m+2r-4)!!}{(n-m)!!}\\
&\qquad\times\cos\frac{(n-m)\pi}{2}\frac{(m+r-1)!}{m!}F_{m+r-1}x^n\\
&=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=0}^{n-2r+2}\frac{(n-m-2)!!}{(n-m-2r+2)!!}\\
&\qquad\times\cos\frac{(n-m-2r+2)\pi}{2}\frac{(m+r-1)!}{m!}F_{m+r-1}x^n\\
&=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\\
&\qquad\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_mx^n\,.
\end{aligned}
$$

Concerning the second term, we have

$$
\begin{aligned}
&\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(1+x^2)^{r+k-1}}f^{(r-k-1)}(x)\\
&=\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{i=0}^{\infty}(-1)^i\binom{i+r+k-2}{r+k-2}x^{2i}\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^m\\
&=\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{l=0}^{\infty}\frac{1}{(r+k-2)!2^{r+k-2}}\frac{(l+2r+2k-4)!!}{l!!}\\
&\qquad\times\cos\frac{l\pi}{2}x^k\sum_{m=0}^{\infty}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^m
\end{aligned}
$$

$$= \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(r+k-2)!2^{r+k-2}}\frac{(n-m+2r+2k-4)!!}{(n-m)!!}$$
$$\times\cos\frac{(n-m)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n$$
$$= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2+2j}^{\infty}\sum_{m=0}^{n-2r+k+2-2j}$$
$$\frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!}\cos\frac{(n-m-2r+k+2-2j)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n\,.$$

Since

$$\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}=0\quad\text{if}\quad m=n-2r+k+2-2j\quad(j=1,2,\dots,k-2)$$

and

$$\cos\frac{(n-m-r+1-2j)\pi}{2}=0\quad\text{if}\quad m=n-2r+k+1-2j\quad(j=0,1,\dots,k-1)\,,$$

this is equal to

$$\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2}^{\infty}\sum_{m=0}^{n-2r+k+2}$$
$$\times\frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!}\cos\frac{(n-m-2r+k+2-2j)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n$$
$$= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}$$
$$\times\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}\cos\frac{(n-m-r+1-2j)\pi}{2}\frac{m!}{(m-r+k+1)!}F_mx^n\,.$$

Since $\cos(\alpha+\pi)=-\cos\alpha$, this is also equal to

$$\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}$$
$$\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_mx^n$$
$$= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}$$
$$\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_mx^n\,.$$

Therefore, the right-hand side of the relation in Theorem 4.2 is

$$\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_m x^n$$
$$+\sum_{k=1}^{r-1}\frac{1}{k(r-k-2)!}\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}$$
$$\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}$$
$$\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n$$
$$=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_m x^n$$
$$+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=1}^{r-2}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n$$
$$+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=r-1}^{n-r+1}\sum_{k=1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n\,.$$

Since for $1\le m\le r-2$ we have

$$\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}$$
$$=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m$$

and for $r-1\le m\le n-r+1$ we have

$$\frac{1}{(r-1)!(r-2)!2^{r-2}}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\frac{m!}{(m-r+1)!}$$
$$+\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}$$
$$=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m\,,$$

the proof is done. □

5. Examples 2

When $r = 2$, Theorem 4.2 is reduced to Theorem 4.1. When $r = 3, 4, 5$, we have the following results as examples.

Theorem 5.1. *For $n \ge 3$, we have*

$$\sum_{\substack{j_1+j_2+j_3=n\\ j_1,j_2,j_3\ge 1}} F_{j_1}F_{j_2}F_{j_3} = \sum_{m=1}^{n-2} \frac{(n+m)(n-m)mF_m}{8}\cos\frac{(n-m-2)\pi}{2}\,,$$

Theorem 5.2. *For $n \ge 4$, we have*

$$\sum_{\substack{j_1+j_2+j_3+j_4=n\\ j_1,j_2,j_3,j_4\ge 1}} F_{j_1}F_{j_2}F_{j_3}F_{j_4}$$
$$= \sum_{m=1}^{n-3} \frac{(n+m+1)(n+m-1)(n-m+1)(n-m-1)mF_m}{4!2^3}\cos\frac{(n-m-3)\pi}{2}\,.$$

Theorem 5.3. *For $n \ge 5$, we have*

$$\sum_{\substack{j_1+\cdots+j_5=n\\ j_1,\dots,j_5\ge 1}} F_{j_1}\cdots F_{j_5}$$
$$= \sum_{m=1}^{n-4} \frac{5(n+m+2)(n+m)(n+m-2)(n-m+2)(n-m)(n-m-2)mF_m}{6!2^6}$$
$$\times \cos\frac{(n-m)\pi}{2}\,.$$

6. Remarks

In [3, Theorem 32.4], it is shown that $\sum_{j=0}^{n} F_j F_{n-j} = h_{2,n}$, where $h_{i,j} = h_{i,j-2} + h_{i,j-1} + h_{i-1,j-1}$ ($i \ge 1$, $j \ge 2$) with $h_{0,j} = 0$ ($j \ge 2$), $h_{j,j} = 1$ ($j \ge 1$) and $h_{i,j} = 0$ ($i > j$). In addition, an explicit form is given by $h_{2,n} = \big((n-1)F_n + 2nF_{n-1}\big)/5$ ([3, (32.13)]). We can show that Theorem 4.1 matches this fact.

In addition, in [3, Theorem 32.4 and (32.14)], it is shown that the left-hand side of (5.1) is equal to $\big((5n^2 - 3n - 2)F_n - 6nF_{n-1}\big)/50$.

Proposition 6.1. *For $n \ge 2$*

$$\sum_{m=1}^{n-1} mF_m \cos\frac{(n-m-1)\pi}{2} = \frac{(n-1)F_n + 2nF_{n-1}}{5}\,.$$

Lemma 6.2. *For $n \geq 0$ and $k \geq 1$ with $k > j$, we have*

$$\sum_{i=0}^{n} F_{ki+j} = \frac{F_{(n+1)k+j} - (-1)^k F_{nk+j} - F_j - (-1)^j F_{k-j}}{L_k - (-1)^k - 1}, \tag{6.1}$$

$$\begin{aligned}\sum_{i=0}^{n} iF_{ki-j} = & \frac{1}{\big(L_k - (-1)^k - 1\big)^2} \Big(nF_{(n+2)k-j} - \big(2(-1)^k n + n + 1\big) F_{(n+1)k-j} \\ & + \big(2(-1)^k (n+1) + n\big) F_{nk-j} - (n+1) F_{(n-1)k-j} \\ & -(-1)^{k+j} F_{k+j} + F_{k-j} + 2(-1)^{k+j} F_j\Big), \end{aligned} \tag{6.2}$$

Proof. (6.1) is Theorem 5.11 in [3]. We shall prove (6.2). Since

$$z + 2z^2 + 3z^3 + \cdots + nz^n = z\frac{d}{dz}(1 + z + z^2 + \cdots + z^n) = \frac{nz^{n+2} - (n+1)z^{n+1} + z}{(z-1)^2},$$

by $\alpha = (1+\sqrt{5})/2$ and $\beta = (1+\sqrt{5})/2$ with $\alpha\beta = -1$, we have

$$\begin{aligned}\sum_{i=1}^{n} iF_{ki-j} &= \sum_{i=1}^{n} i\frac{\alpha^{ki-j} - \beta^{ki-j}}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha^j}\frac{n(\alpha^k)^{n+2} - (n+1)(\alpha^k)^{n+1} + \alpha}{(\alpha^k - 1)^2} - \frac{1}{\beta^j}\frac{n(\beta^k)^{n+2} - (n+1)(\beta^k)^{n+1} + \beta}{(\beta^k - 1)^2}\right) \\ &= \frac{1}{\sqrt{5}\big((\alpha\beta)^k - (\alpha^k + \beta^k) + 1\big)^2}\big(n(\alpha^{nk-j} - \beta^{nk-j}) - (n+1)(\alpha^{(n-1)k-j} - \beta^{(n-1)k-j}) \\ &\quad - (-1)^k(\alpha^k\beta^{-j} - \beta^k\alpha^{-j}) - 2n(-1)^k(\alpha^{(n+1)k-j} \\ &\quad - \beta^{(n+1)k-j}) + 2(-1)^k(n+1)(\alpha^{nk-j} - \beta^{nk-j}) - 2(-1)^k(\alpha^{-j} - \beta^{-j}) \\ &\quad + n(\alpha^{(n+2)k-j} - \beta^{(n+2)k-j}) - (n+1)(\alpha^{(n+1)k-j} - \beta^{(n+1)k-j}) + (\alpha^{k-j} - \beta^{k-j})\big) \\ &= \frac{1}{\big(L_k - (-1)^k - 1\big)^2}\Big(nF_{(n+2)k-j} - \big(2(-1)^k n + n + 1\big)F_{(n+1)k-j} \\ &\quad + \big(2(-1)^k(n+1) + n\big)F_{nk-j} - (n+1)F_{(n-1)k-j} \\ &\quad - (-1)^{k+j}F_{k+j} + F_{k-j} + 2(-1)^{k+j}F_j\Big). \end{aligned}$$

Here, we used the fact $F_{-j} = (-1)^{j-1}F_j$ $(j \geq 1)$. □

Proof of Proposition 6.1. Let $n = 4k$. Other cases $n \not\equiv 0 \pmod 4$ can be proven similarly. By Lemma 6.2

$$\begin{aligned}
&\sum_{m=1}^{n-1} mF_m \cos\frac{(n-m-1)\pi}{2} \\
&= -\sum_{l=1}^{k}(4l-3)F_{4l-3} + \sum_{l=1}^{k}(4l-1)F_{4l-1} \\
&= 4\sum_{l=1}^{k} lF_{4l-2} + \sum_{l=1}^{k} F_{4l-5} \\
&= \frac{4}{25}\big(kF_{4k+6} - (3k+1)F_{4k+2} + (3k+2)F_{4k-2} - (k+1)F_{4k-6} - 5\big) \\
&\quad + \frac{1}{5}(F_{4k-1} - F_{4k-5} + 4) \\
&= \frac{(4k-1)F_{4k} + 2F_{4k-1}}{5} = \frac{(n-1)F_n + 2F_{n-1}}{5}.
\end{aligned}$$

□

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School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China
E-mail address: komatsu@whu.edu.cn

Department of Mathematics FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic
E-mail address: zuzana.masakova@fjfi.cvut.cz

Department of Mathematics FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic
E-mail address: edita.pelantova@fjfi.cvut.cz