A KNAPSACK-LIKE CODE USING RECURRENCE SEQUENCE REPRESENTATIONS

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ABSTRACT. We had recently shown that every positive integer can be represented uniquely using a recurrence sequence, when certain restrictions on the digit strings are satisfied. We present the details of how such representations can be used to build a knapsack-like public key cryptosystem. We also present new disguising methods, and provide arguments for the security of the code against known methods of attack.

1. INTRODUCTION

One of the first public key cryptosystems was the traditional knapsack code proposed by Merkle and Hellman [11]. This code has the advantage of fast encoding and decoding. Also, in more recent developments, it has been shown that a quantum computer could make factoring large numbers fast enough to make the RSA code insecure [15, 10]. However, it appears that quantum computers would still struggle to make knapsack problems fast to solve [1].

Unfortunately, the traditional knapsack code was broken by two different approaches – by reversing the disguising steps [14] and by a direct attack using lattice-based approaches [8]. We describe a new type of knapsack-like code along with new disguising techniques, which make the code resistant to both these classes of attacks. We first give a brief description of the traditional knapsack code and its weaknesses.

1.1. **The traditional knapsack code.** The plain text message is assumed to be an integer M, $0 \leq M < 2^n$. We consider the representation of M in base 2:

$$M = \sum_{i=0}^{n-1} \epsilon_i 2^i, \ 0 \leq \epsilon_i \leq 1. \tag{1}$$

The creator of the code chooses a secret, super increasing sequence $\{s_i\}$, i.e., where $s_i > s_{i-1} + \cdots + s_0$. The secret s_i are then disguised by one or more modular multiplications of the form

$$w_i = k s_i \,(\mathrm{mod}\ m), \tag{2}$$

where k and m are kept secret. The w_i are made public. The sender of the message M computes

$$T = \sum_{i=0}^{n-1} \epsilon_i w_i. \tag{3}$$

The encoded message T is then sent over a possibly insecure channel. The hope was that only the creator of the code, who knows k, m, and the s_i, can solve equation (3) for the coefficients ϵ_i. In particular, the disguising step given in equation (2) is easily reversed.

A KNAPSACK-LIKE CODE USING RECURRENCE SEQUENCE REPRESENTATIONS

Shamir was able to break this code by calculating k and m and thus reversing the disguising [14]. Although the w_i appear random, the fact that the s_i are superincreasing can be exploited to yield enough clues to determine k and m, or at least equally useful alternative values.

The other attack, which has received the most attention, tries to solve equation (3) directly using lattice-based approaches [8, 2]. In essence, the solutions to equation (3) correspond to a lattice of vectors. Basis reduction algorithms are efficient methods to find short vectors in, and short bases of, lattices [9, 7]. Although not guaranteed to do so, these approaches often find a shortest vector in the given lattice [3]. A weakness in using the base-2 representation of the message M is that the corresponding vector $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_0 & \epsilon_1 & \cdots & \epsilon_{n-1} \end{bmatrix}^T$ from equations (1) and (3) is likely to be the unique shortest vector in an appropriately defined basis.

For example, if $n = 1000$, the size of M is roughly 2^{1000}. After using the s_i and disguising to create the w_i (using equation (2)), the target sum T might be of size 2^{1030} (depending on the number of disguising steps used). The expected length of the true decoding vector $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_0 & \epsilon_1 & \cdots & \epsilon_{999} \end{bmatrix}^T$ is 500 (for simplicity, we talk about the *square* of the length here). The number of 0–1 1000-vectors of length less than 500 is roughly 2^{999}. Hence, the probability of any one of them equaling T is 2^{-31}. In practice, approaches using basis reduction may find vectors with some $\epsilon_i = -1$ or $\epsilon_i = 2$, and so on. We may penalize such solutions in these approaches, but cannot forbid them. But allowing such ϵ_i values tend to produce longer vectors. Thus it is indeed likely that the shortest vector in the lattice is the true decoding vector. In our new knapsack-like code, we try to prevent such lattice-based approaches from being effective.

2. A CODE USING RECURRENCE SEQUENCE REPRESENTATIONS

Hamlin and Webb recently presented a description of how to find a unique representation of any positive integer using a recurrence sequence $\{u_i\}$ as a base [4]. We now show how to create a knapsack-like public key code using such representations. We illustrate our construction on a small example, and refer the reader to the above paper [4] for the complete proofs of all assertions.

2.1. **An example code.** Let u_i satisfy the recurrence $u_{i+5} = u_{i+4} + u_{i+2} + 2u_{i+1} + 7u_i$ (the initial values are not vital, and we could take them to be the standard ones, i.e., $u_0 = 1, u_1 = 1, u_2 = 1, u_3 = 2$, and $u_4 = 4$). The signature is $\mathcal{S} = 10127$. The representation of any positive integer M is of the form

$$M = \sum_{i=0}^{n-1} d_i u_i, \tag{4}$$

where the string of digits $d_{n-1}d_{n-2}\ldots d_1 d_0$ must be composed of blocks of digits which are lexicographically smaller than $\mathcal{S}$. In this case, the allowed blocks of digits are $0, 100, 1010, 1011,$ $10120, 10121, 10122, 10123, 10124, 10125, 10126$. Hence, for instance, $1011|1023|0|1010$ is a legitimate string, but $1010|110123100$ is not. Notice that no allowed block begins with 11.

We now illustrate how to calculate this type of representation of any number M using a greedy approach. Although this calculation is straightforward, the fact that makes this code harder to model for the cryptanalyst is that so many strings of digits are not allowed in the representations, even though the strings appear similar to the allowed ones, and both classes of strings use digits of the same size.

2.1.1. *Computing the representation of M*. An easy way to calculate the representation of any number M in the recurrence sequence $\{u_i\}$ is to calculate the augmented sequence $\{v_{j,i}\}$ for $1 \leq j \leq 10$, $0 \leq i \leq n$ given by

$$v_{1,i} = u_i,$$
$$v_{2,i} = u_i + u_{i-2},$$
$$v_{3,i} = u_i + u_{i-2} + u_{i-3},$$
$$v_{4,i} = u_i + u_{i-2} + 2u_{i-3},$$
$$v_{5,i} = u_i + u_{i-2} + 2u_{i-3} + u_{i-4},$$
$$\vdots$$
$$v_{10,i} = u_i + u_{i-2} + 2u_{i-3} + 6u_{i-4}.$$

The $v_{j,i}$ correspond to the allowed blocks of digits. In our example, the $v_{j,i}$ occur in groups of size 10. In other examples, the groups could be much larger.

The correct expression for M is found simply by using the greedy algorithm on the $v_{j,i}$, and converting the sum into an expression in the u_i. The $v_{j,i}$ could be calculated and stored, or calculated from the u_i as needed.

We explore the memory requirements for storing all the $\{v_{j,i}\}$. The principal eigenvalue of the sequence $\{u_i\}$ is $\alpha \approx 1.9754$. We may assume that u_i is roughly α^i, or is close to 2^i. After disguising, the public weights w_i will be approximately 2^{n+40}, and the target T approximately 2^{n+50}. In other words, these quantities require 40–50 extra bits of memory to represent. If $n = 1000$, the memory required for the weights w_i is roughly $1,040,000$ bits (or 130 kilobytes). The memory required to store all the $v_{j,i}$ is hence 1.3 megabytes. Even if n is much larger, the memory needed for the w_i is negligible.

2.2. **Encryption and decryption.** Let $\{u_i\}$ be a recurrence sequence that satisfies the following recurrence equation.

$$u_i = a_1 u_{i-1} + a_2 u_{i-2} + \cdots + a_h u_{i-h}, \tag{5}$$

where $a_1 > 0$ and all $a_i \geq 0$. The string $\mathcal{S} = a_1 a_2 \cdots a_h$ is its signature, and we let $A = a_1 + a_2 + \cdots + a_h$. Every natural number N has a unique representation in the form of equation (4), where the digits are composed of blocks that are lexicographically smaller than $\mathcal{S}$. Including the zero block, there are A such blocks. The auxiliary sequence $\{v_{j,i}\}$ is constructed as linear combinations of the u_i with coefficients same as the blocks other than the zero block. Hence there are $A - 1$ of the $v_{j,i}$ in each group. The total number of $v_{j,i}$ numbers is hence, $(A-1)n$ if there are n of the u_i.

The creator of the code chooses a secret sequence $\{s_i\}$ which has the property that $s_{i+1} > s_i (u_{i+1}/u_i)$ for all i. This property replaces the condition of $\{s_i\}$ being superincreasing as used in the traditional knapsack code.

The s_i are then disguised by any invertible mapping, some of which we describe below. The resulting quantities w_i are the public weights. If M is the original plaintext message, the user of the code expresses

$$M = \sum_{i=0}^{n-1} d_i u_i = \sum_{i=0}^{n-1} \epsilon_{j,i} \, v_{j,i}, \tag{6}$$

and computes

$$T = \sum_{i=0}^{n-1} d_i w_i = \sum_{i=0}^{n-1} \epsilon_{j,i} \, y_{j,i}, \tag{7}$$

which is the transmitted message.

Since the mappings used for disguising are invertible by the code's creator, one can calculate

$$N = \sum_{i=0}^{n-1} d_i s_i = \sum_{i=0}^{n-1} \epsilon_{j,i} \, t_{j,i}. \tag{8}$$

We must show that one can solve equation (8) for the *same* digits d_i as appearing in equation (6). The $t_{j,i}$ and $y_{j,i}$ are combinations of the s_i and w_i, respectively, in the same way as $v_{j,i}$ are combinations of the u_i. Also, each $\epsilon_{j,i} = 0$ or 1, and for each i, at most one $\epsilon_{j,i} = 1$.

In the greedy algorithm to express N using the $t_{j,i}$, we subtract the largest possible $t_{j,i}$ from N at each step, and repeat the process on the remainder. If the correct $t_{j,i}$ have been used previously, we have an equation at each step that is essentially of the same form as equation (8). That is, we know the number N' and that

$$N' = \sum_{i=0}^{k} \epsilon_{j,i} \, t_{j,i}, \tag{9}$$

where the corresponding number

$$M' = \sum_{i=0}^{k} \epsilon_{j,i} \, v_{j,i} \tag{10}$$

is used when expressing M. We rewrite equations (9) and (10) as

$$N' = t_{j_1} + t_{j_2} + t_{j_3} + \cdots, \quad \text{and} \tag{11}$$

$$M' = v_{j_1} + v_{j_2} + v_{j_3} + \cdots, \tag{12}$$

where $j_1 > j_2 > j_3 > \cdots$. In other words, we include only the terms for which $\epsilon_{j,i} = 1$.

From equation (11), we get $t_{j_1} \leq N'$. Hence the greedy algorithm will use t_{j_1} unless the next larger number in the sequence $t_{j_1+1} \leq N'$, in which case t_{j_1+1} would be used. By the definition, $t_{j_1+1} = t_{j_1} + s_q$ for some s_q. Also, $v_{j_1+1} = v_{j_1} + u_q$, but v_{j_1+1} was not used in expressing M'. Hence it must be true that $v_{j_1+1} = v_{j_1} + u_q > M'$, whereas $t_{j_1} + s_q \leq N'$.

Now we replace the t_j and v_j by their corresponding combinations of the s_j and u_j, respectively, as follows.

$$s_q \leq N' - t_{j_1} \ = \ t_{j_2} + t_{j_3} + \cdots \ = \ b_1 s_{i_1} + b_2 s_{i_2} + \cdots, \quad \text{and} \tag{13}$$

$$u_q > M' - v_{j_1} = v_{j_2} + v_{j_3} + \cdots = b_1 u_{i_1} + b_2 u_{i_2} + \cdots. \tag{14}$$

Since q is larger than any of the i_r in equations (13) and (14), and since the s_i were chosen so that $s_i/s_{i+1} < u_i/u_{i+1}$, from equations (13) and (14) we have

$$1 \leq b_1(s_{i_1}/s_q) + b_2(s_{i_2}/s_q) + \cdots < b_1(u_{i_1}/u_q) + b_2(u_{i_2}/u_q) + \cdots < 1,$$

which is a contradiction. Hence the greedy algorithm will indeed use t_{j_1}.

3. Disguising Methods

As described above, the plaintext message M can be expressed either as $\sum d_i u_i$ or as $\sum \epsilon_{j,i} v_{j,i}$ where $\epsilon_{j,i} = 0$ or 1, and can be calculated using a greedy algorithm. The sequence $\{s_i\}$ is chosen with the related auxiliary sequence $\{t_{j,i}\}$, which corresponds to the $v_{j,i}$. Then, if M is expressed as in equation (6) and N as in equation (8), the creator of the code can calculate the $\epsilon_{j,i}$ from knowing N, and can thus compute M. A disguising method that maps s_i into w_i is invertible if given T as expressed in equation (7), one can compute the number N. The $y_{j,i}$ are defined in terms of the w_i, and in the same way $t_{j,i}$ are defined in terms of s_i and $v_{j,i}$ in terms of u_i.

Let $E = \max \sum \epsilon_{j,i}$, where the maximum is taken over all expressions of possible messages M. If n is the number of the u_i then $E \leq n$ since using a greedy algorithm as described, at most one $v_{j,i}$ in each of n groups is used. With more careful analysis we can show that E is much smaller than n, but that result will not be critical for our analysis.

For example, in the usual modular multiplication, $w_i \equiv c s_i \pmod{m}$ or $s_i \equiv \bar{c} w_i \pmod{m}$. Then

$$\bar{c} T \equiv \sum_i d_i \bar{c} w_i \equiv \sum_i d_i s_i \equiv N \pmod{m}.$$

The number N is uniquely determined if $N < m$, which is true if $m > E \max\{t_{j,i}\}$. Thus, it suffices to take $m > E s_n$. Since the w_i are defined modulo m, the w_i are larger than the s_i by a factor of $E s_n$. As such, the w_i require $\log_2(E s_n)$ bits to express in base 2. This expansion in the size of the disguised weights by $\log_2(E)$ bits turns out to be similar in each stage of the disguising.

As described earlier in Section 1.1, the nature of modular multiplication and the fact that some of the s_i are very small give the cryptanalyst a way to possibly compute the parameters c and m. We now describe some alternative disguising methods and indicate why they could not be reversed by the cryptanalyst.

First, perform an ordinary modular multiplication so that all the obtained values are of roughly the same size. Take any collection of pairwise prime moduli. For simplicity, we could use distinct primes p_k for $1 \leq k \leq r$. Replace the weight w_i by the r-vector $\mathbf{W}_i = \begin{bmatrix} w_i \pmod{p_1} & w_i \pmod{p_2} & \cdots & w_i \pmod{p_r} \end{bmatrix}^T$. Then the jth component of

$$\sum_i d_i \mathbf{W}_i \equiv \sum_i d_i w_i \pmod{p_j}. \tag{15}$$

Hence we can use the Chinese Remainder Theorem to compute $\sum_i d_i w_i \pmod{p_1 \ldots p_k}$, and so the numerical value of $\sum_i d_i w_i$ is determined if $p_1 \ldots p_k > AE \max\{w_i\}$. Again, the number of bits needed to express the vector $\mathbf{W}_i$ is $\log_2(AE \max\{w_i\})$ more than the number needed to express the w_i.

We can do two stages of this type of mapping, the first one with large moduli m_1 and m_2 resulting in vectors of the form $[w_i' \ w_i'']$ for the original weight w_i. The second stage uses a large number of small primes separately on w_i' and w_i''. These primes could be the same or different for w_i' and w_i''. In fact, we could simply take $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on.

Since each of the two residues modulo m_1 and m_2 are disguised in this way, we have two lists of residues modulo 2 and 3 and 5, and so no. The creator of the code can choose a secret permutation of these residues. When there are k primes $p_1, \ldots, p_k$, each weight is a vector of dimension $2k$. A cryptanalyst could easily see from the size of the components whether a given component is modulo 2 or modulo 3, and so on. But he cannot know which of the two

residues modulo p_i came from the m_1 branch and which from the m_2 branch of the disguising method. There are 2^k possible choices here. Although it is easy to use the Chinese Remainder Theorem on any such choice, making even one incorrect choice will produce incorrect residues modulo m_1 and m_2. Further, the values m_1 and m_2 are not known to the cryptanalyst. Thus the security of the disguising rests on the large number of permutations of the residues modulo p_i, and not on the difficulty of solving a particular type of calculation.

This method can be combined with other mappings as well. Another strong candidate is using modular multiplications in the rings of algebraic integers [13]. This step also turns each ordinary integer weight into a vector of dimension k, where k is the degree of the algebraic extension used. A sequence of such steps can be represented as a tree. Each sequence of steps corresponds to a different tree, a different set of parameters, and a different system of equations describing these parameters. Even if we assume that a cryptanalyst could obtain information about the disguising mapping from a system of such equations, he does not even know what system of equations is the correct one to solve. The number of such possibilities could be made as large as we wish. Note that the final public weight vector consists of the leaves of the corresponding tree, which could also be secretly permuted. Thus the cryptanalyst does not know the correct permutation nor the correct tree model to use. Any incorrect choice results in a nondecoding result, as a consequence of the Chinese Remainder Theorem.

4. Cryptanalysis

The disguising methods we just described make an attack that hopes to reverse the disguising steps quite unlikely to succeed. But even for the traditional knapsack code, attacks that try to solve equation (3) directly using basis reduction algorithms [8, 2] have posed the greater threat to security.

Solving equation (3) along with the constraints that $\epsilon_i = 0$ or 1 could be modeled exactly using integer optimization methods. While directly solving these instances with just this single constraint could be difficult even for moderate values of n, basis reduction-based reformulations could be more effective [6]. But adding more constraints makes the integer optimization instances increasingly hard to solve even when we have only a few hundred weights, e.g., see the recent work on basis reduction-based methods to solve market split problems [16]. Solving equation (7) with very complex constraints and thousands of weights appears impossible using current methods.

Direct basis reduction-based approaches, as opposed to integer optimization approaches, can handle much larger problems. But these methods cannot impose strict adherence to the constraints as the integer optimization models do. Instead, these algorithms find short vectors in appropriately defined lattices, which correspond to solutions of equations like (3) and (7). In the case of equation (3), the shortest vector in the lattice is the desired vector of ϵ_i, even though the basis reduction-based methods are not guaranteed to find this particular vector. Indeed, the *shortest vector problem* (SVP) and the closely related *closest vector problem* (CVP) are known to be hard problems. CVP is known to be NP-complete, and so is SVP under randomized reductions [12]. Still, such algorithms are often successful in practice to solve the problem instances exactly [3].

Most basis reduction-based approaches on the default knapsack code in equation (3) start by defining an appropriate lattice in which the shortest vector corresponds to the correct decoding message. For the subset sum problem with weights w_i and the target sum T, Coster

et al. [2] consider the lattice generated by

$$\mathcal{L} = \begin{bmatrix} c\,\mathbf{w}^T & c\,T \\ 2I & \mathbf{1} \end{bmatrix}, \tag{16}$$

where $\mathbf{w}$ is the vector of weights w_i, I is the $n \times n$ identity matrix, and $\mathbf{1}$ is the n-vector of ones. When the multiplier c is chosen large enough, the vector $\boldsymbol{\epsilon}$ corresponding to the correct decoding will generate the shortest vector in this lattice by multiplying the first n columns of $\mathcal{L}$ by $\boldsymbol{\epsilon}$ and the last column by -1. This shortest vector has a length of $\sqrt{n}$. To locate this shortest vector, one tries to find short(est) vectors in the lattice $\mathcal{L}$. While it is not guaranteed to find the correct decoding vector in every case, these algorithms succeed with high probability when the *density* of the knapsack, defined as $n/(\max_i \log(w_i))$, is small. When n is of the order of a few hundreds, these methods have been shown to be effective in finding the correct decoding [5].

To see how these methods might be applied to our code, we examine the example described in Section 2.1. The cryptanalyst has the choice of trying to solve either

$$T = \sum_{i=0}^{n-1} d_i w_i \qquad \text{or} \qquad T = \sum_{i=0}^{n-1} \epsilon_{j,i}\, y_{j,i}. \tag{(7) revisited}$$

We examine both possibilities by first calculating the expected lengths of the vectors $\mathbf{d} = [d_0 \;\cdots\; d_{n-1}]^T$ of dimension $n = 1000$ and the vector $\boldsymbol{\epsilon}$ of $\epsilon_{i,j}$ values of length $10n = 10^4$. It is not clear how one would add constraints forbidding the particular substrings of digits that are not permitted in $\mathbf{d}$ in the former approach. While one could potentially model all constraints forbidding nonallowed combinations of $\epsilon_{i,j}$ in the latter approach, this step would produce a candidate lattice $\mathcal{L}$ in which the single knapsack row is replaced by a substantially large number of simultaneous linear Diophantine equations. Hence the original basis reduction-based methods will struggle to find the correct decoding vector in this case as well. We could add one further level of difficulty in modeling the correct lattice $\mathcal{L}$ for the latter approach using $\epsilon_{i,j}$. We could slightly alter M by adding a small number $\bar{M}$, or by using $2M$ or $3M$ instead of M, so that the expected length of the decoding vector becomes longer. The value $\bar{M}$, or the multiplier, is sent in the clear, and should not affect the security of the code.

Further, since the first few groups of the $v_{j,i}$ do not have the same number of elements, it is convenient to start with, say, u_{20} instead of u_0. In the encoding process, the remaining values smaller than u_{20} are also sent in the clear. This modification also makes the choice of the sequence of s_i easier to make, without increasing the size of the final public weights much. Sending these small values in the clear does not change the efficiency of the code significantly.

In the encoding procedure, suppose at some stage we have the value M' as in equation (10). If M' falls randomly in the interval $[u_k, u_{k+1})$, we first calculate the probability that $v_{j,k} \leq M' < v_{j+1,k}$. Then we calculate the probability that $M' - v_{j,k}$ falls in a particular smaller interval $[u_\ell, u_{\ell+1})$. Recall that the associated principal eigenvalue is $\alpha \approx 1.9754$, and we may approximate u_k by α^k for large k. There are 10 subintervals to consider. Letting $v_{j,k} = v_j$ temporarily for ease of notation, the intervals are $I_1 = [v_1,\, v_2 = v_1 + u_{k-2})$, $I_2 = [v_2,\, v_2 + u_{k-3})$, $I_3 = [v_3,\, v_3 + u_{k-3})$, and $I_j = [v_j,\, v_j + u_{k-4})$ for $4 \leq j \leq 10$, where $v_{10} + u_{k-4} = u_{k+1}$.

If $h \leq k-5$, then $M' - v_{j,k} \in [u_h, u_{h+1})$ can occur in any of the 10 subintervals. If $h = k-4$, then $M' - v_{j,k} \in [u_{k-4}, u_{k-3})$ can occur only in intervals I_1, I_2, or I_3, and $M' - v_{j,k} \in [u_{k-3}, u_{k-2})$ can occur only in I_1. Hence the probability that $M' - u_k \in [u_{k-5}, u_{k-4})$ is $10(u_{k-4} - u_{k-5})/(u_{k+1} - u_k) \approx 10\alpha^{-5}$, while the probability that $M' - u_k \in [u_{k-4}, u_{k-3})$ is $3(u_{k-3} - u_{k-4})/(u_{k+1} - u_k) \approx 3\alpha^{-4}$. If $M' \in [u_k, u_{k+1})$ and $M' - u_k \in [u_h, u_{h+1})$, then there

are $k - h - 1$ groups of the $v_{j,i}$ that are skipped, i.e., they are not present in the representation of M'. Then the expected number of skipped groups between each pair that does appear is approximately

$$10 \sum_{h=0}^{k-5} \alpha^{h-k} (k - h - 1) + 3\alpha^{-4}(3) + \alpha^{-3}(2)$$

$$= 10 \sum_{j=5}^{k} \alpha^{-j} (j - 1) + 9\alpha^{-4} + 2\alpha^{-3}$$

$$\approx 10 \left(\alpha^{-3} + 3(\alpha - 1)\alpha^{-4} / (\alpha - 1)^2 \right) + 9\alpha^{-4} + 2\alpha^{-3}$$

$$\approx 3.383 \ + \ 0.591 \ + \ 0.259 \ = \ 4.233.$$

Therefore the expected number of groups represented when expressing M is $n / 5.233$, or 191 when $n = 1000$.

In this example, $M \approx \alpha^{1000} \approx 5.5 \times 10^{295} \approx 2^{982}$. After the disguising steps described above, the size of the target sum T in number of bits needed is about 1030. There are then $2^{1030} \approx 10^{310}$ possible target objects or vectors. Even if we use extra disguising steps, the number of target objects might be as large as 10^{325}.

We expect the representation of M to use some $v_{i,j}$ from 191 of the 1000 groups. As described above, we can easily alter M slightly to make sure at least 191 groups are used, i.e., the vector corresponding to the correct decoding has length at least 191. If we look at shorter vectors, say of length 180, consisting of one $v_{i,j}$ from 180 different groups, there are

$$\left(\begin{array}{c} 1000 \\ 180 \end{array} \right) 10^{180} \ \approx \ 10^{383}$$

such vectors. Thus we expect $10^{383-325} = 10^{58}$ of these vectors to yield the same T. There are many even shorter vectors that yield T. If a basis reduction-based method finds the shortest vector yielding T, it will not be the one needed to decode T. If the method finds some short vector at random, the chance that it is the correct decoding vector would be much smaller than 10^{-58}.

Now suppose the cryptanalyst tries to solve equation (7) directly for the vector of digits $\mathbf{d}$ instead. The candidate vectors have dimension 1000 now, but the entries may be as large as 6.

To compute the expected length of $\mathbf{d}$, we need to calculate the expected value of each digit. By the calculations described above, we expect 191 of the 10 blocks $100, 1010, \ldots, 10126$ to appear. The jth block is used if M is in the interval I_j described above. Hence the block 100 is used with probability $u_{k-2} / (u_{k+1} - u_k) \approx \alpha^{-2}/(\alpha - 1)$, 1010 and 1011 appear with probability $\alpha^{-3}/(\alpha - 1)$, and the other 7 blocks with probability $\alpha^{-4}/(\alpha - 1)$.

Among the 1000 digits d_i, we expect roughly 475 zeros, 370 ones, 103 twos, and 13 each of $3, 4, 5$, and 6. The expected length of the vector is 1900. There are more than 10^{441} vectors of length 1550 consisting of 200 zeros, 500 ones, 150 twos and 50 threes. There exist even more shorter vectors. Since there are only 10^{325} target vectors, we expect many shorter vectors to correspond to the same value T. Thus, unless it is possible to force the basis reduction-based algorithms to exclude vectors that do not correspond to allowable strings of digits, these methods will yield shorter vectors that do not correctly decode the encrypted message T. Integer optimization-based methods could exclude nonallowed vectors, but they could not handle problems of this size.

The example code presented above is actually smaller and simpler than one that would be suggested for use. Indeed, one could take n as large as 10^5 without needing too large an amount of memory, and A as large as 100 or 1000. This setting would require the cryptanalyst to solve problems with 10^8 vectors if he used the version of equation (7) with $y_{j,i}$.

5. Discussion

We have described how to use recurrence sequence representations to create a knapsack-like public key code. There is considerable freedom in the choice for the encoding sequence $\{u_i\}$, as well as its order h, the size and pattern of the coefficients a_j, and the number of terms n. Larger orders and coefficients create more false short vectors and hence increase the complexity of cryptanalysis. However, they also increase the size of u_i and the weights w_i. Questions about optimal choices for these parameters remain for further study.

Another relevant question is whether specific number theoretic properties of such representations could be used to make this code even more secure. In particular, security against attacks targeting the representation using $\epsilon_{i,j}$ relies on the size of the instances, which could not be handled by state of the art methods. But could we make such attacks less feasible, or even impossible to mount, using inherent properties of the sequence used for the representation?

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