ON THE EXISTENCE OF VAN DER WAERDEN TYPE NUMBERS FOR LINEAR RECURRENCE SEQUENCES WITH CONSTANT COEFFICIENTS

GÁBOR NYUL AND BETTINA RAUF

Abstract. We prove some existence and non-existence results on van der Waerden type numbers for linear recurrence sequences with constant positive integer coefficients.

1. Introduction

Let X be a non-empty set and C a set of colors with r elements where r is a positive integer. Then a function $\chi : X \to C$ is called an r-coloring of X. A non-empty subset $Y \subseteq X$ is monochromatic under this coloring if χ is constant on Y. Theorems of Ramsey theory typically say that if X is large enough in some sense, then for any r-coloring of X, there exists a monochromatic subset with some given properties.

Ramsey type results in the area of number theory usually deal with colorings of the positive integers. One of the oldest Ramsey type theorems is due to I. Schur [12]. It states that for any integer $r \geq 1$, there exists a least positive integer $s(r)$ such that for any r-coloring of $[1, s(r)] \cap \mathbb{Z}$, there exists a monochromatic solution of the equation $x + y = z$.

Schur's result was later generalized to systems of linear equations. A homogeneous system $\mathcal{E}$ of linear equations with integer coefficients is said to be regular if for any integer $r \geq 1$, there exists a least positive integer $g(\mathcal{E}, r)$ such that for any r-coloring of $[1, g(\mathcal{E}, r)] \cap \mathbb{Z}$, there exists a monochromatic solution of $\mathcal{E}$. Let C be an $m \times n$ integer matrix with $n \geq 2$ and denote its jth column by c_j ($j = 1, \dots, n$). R. Rado [11] proved that the system $Cx = 0$ is regular if and only if C satisfies the so-called columns condition: there exists a partition $\{I_1, \dots, I_t\}$ of $\{1, \dots, n\}$ such that $\sum\limits_{j \in I_1} c_j = 0$ and $\sum\limits_{j \in I_i} c_j$ is a linear combination of columns c_k ($k \in I_1 \cup \dots \cup I_{i-1}$) for $i = 2, \dots, t$. For a single equation $\sum\limits_{i=1}^{n} c_i x_i = 0$ with coefficients $c_i \in \mathbb{Z} \setminus \{0\}$ ($i = 1, \dots, n$), this means equivalently that there exists $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$ such that $\sum\limits_{i \in I} c_i = 0$.

Instead of monochromatic solutions of Diophantine equations, one can search for monochromatic sequences. The classical theorem of B. L. van der Waerden [13] asserts that for any integers $k \geq 2$ and $r \geq 1$, there exists a least positive integer $w(k, r)$ such that for any r-coloring of $[1, w(k, r)] \cap \mathbb{Z}$, there exists a monochromatic, strictly increasing finite arithmetic progression of length k.

Research was partially supported by the European Union and the European Social Fund through project Supercomputer, the national virtual lab (grant no.: TÁMOP-4.2.2.C-11/1/KONV-2012-0010), and by Grant 100339 from the Hungarian Scientific Research Fund.

The similar problem arises when we replace arithmetic progressions with another family of sequences. Let $\mathcal{S}$ be a non-empty set of sequences of positive integers. On a finite sequence of $\mathcal{S}$ of length k, we mean the first k elements of a sequence from $\mathcal{S}$. For integers $k \geq 2$ and $r \geq 1$, let $w(\mathcal{S}, k, r)$ be the least positive integer if it exists, such that for any r-coloring of $[1, w(\mathcal{S}, k, r)] \cap \mathbb{Z}$, there exists a monochromatic finite sequence of $\mathcal{S}$ of length k. Obviously, $w(\mathcal{S}, k, 1)$ exists for any $k \geq 2$. Moreover, if $k \geq k' \geq 2$, $r \geq r' \geq 1$ and $w(\mathcal{S}, k, r)$ exists, then $w(\mathcal{S}, k', r)$ and $w(\mathcal{S}, k, r')$ also exist and $w(\mathcal{S}, k', r) \leq w(\mathcal{S}, k, r)$ and $w(\mathcal{S}, k, r') \leq w(\mathcal{S}, k, r)$.

For example, for the family of strictly increasing geometric progressions, the existence of van der Waerden type numbers can be deduced from van der Waerden's theorem. Several other families of sequences are investigated, for example, in [9].

However, there are only a few results for linear recurrence sequences. Since strictly increasing arithmetic progressions are exactly those sequences $(x_i)_{i=1}^{\infty}$ which satisfy the recurrence relation $x_i = 2x_{i-1} - x_{i-2}$ $(i \geq 3)$ and the inequality $x_1 < x_2$ for the initial values, van der Waerden's original theorem can be viewed as the first such result.

Denote by $\mathcal{F}$ the set of sequences $(x_i)_{i=1}^{\infty}$ of positive integers which satisfy the Fibonacci recurrence $x_i = x_{i-1} + x_{i-2}$ $(i \geq 3)$. Schur's Theorem guarantees the existence of $w(\mathcal{F}, 3, r) = s(r)$ for any $r \geq 1$.

B. M. Landman [8] (see also [9, Section 3.6]) gave an upper bound on van der Waerden type numbers in the case of $r = 2$, among others for the family of sequences satisfying the linear recurrence $x_i = b_i x_{i-1} + (1 - b_i)x_{i-2}$ $(i \geq 3)$, where $x_1 < x_2$ and $b_i \in [2, +\infty[\cap \mathbb{Z}$ are non-fixed coefficients. His results were recently generalized by G. Nyul and B. Rauf [10].

Second order linear recurrence sequences with constant positive integer coefficients were studied by H. Harborth and S. Maasberg [5, 6], with a special emphasis on the Fibonacci recurrence. In [6] they proved that for $k \geq 3$ and $r \geq 2$, $w(\mathcal{F}, k, r)$ exists if and only if $k = 3$ or $k = 4$. Moreover, if $\widetilde{\mathcal{F}} = \{(x_i)_{i=1}^{\infty} \in \mathcal{F} \mid x_1 \leq x_2\}$, then $w(\widetilde{\mathcal{F}}, k, r)$ exists if and only if $k = 3$. This latter assertion was rediscovered by H. Ardal, D. S. Gunderson, V. Jungić, B. M. Landman and K. Williamson [2]. We notice that H. Harborth, S. Maasberg [7] and H. Ardal, Z. Dvořák, V. Jungić, T. Kaiser [1] investigated similar problems for second order linear recurrence sequences with rational coefficients, but these results do not fit to our case, since only the first few elements of these sequences are certainly positive integers.

However, higher order linear recurrence sequences have not been studied yet. In Section 2, our purpose is to give some existence and non-existence theorems for linear recurrence sequences with constant positive integer coefficients of any order. After these general results, as a special case, we turn our attention to multibonacci sequences in Section 3.

2. Linear Recurrence Sequences with Positive Integer Coefficients

In the sequel, we study linear recurrence sequences, where the coefficients in the recurrence equation are fixed positive integers.

Let $s \geq 2$ and $a_1, \ldots, a_s$ be positive integers. Denote by $\mathcal{R}_{a_1,\ldots,a_s}$ the family of those sequences $(x_i)_{i=1}^{\infty}$ which consist of positive integers and satisfy the linear recurrence $x_i = a_1 x_{i-s} + \cdots + a_s x_{i-1}$ $(i \geq s + 1)$ of order s.

We notice that for $k \leq s$ and arbitrary $r \geq 1$, the first k terms of the sequence from $\mathcal{R}_{a_1,\ldots,a_s}$ with initial values $x_1 = \cdots = x_s = 1$ give a monochromatic finite sequence of $\mathcal{R}_{a_1,\ldots,a_s}$ of length k, independently from the choice of the r-coloring. Therefore, it is enough to consider the case when $k \geq s + 1$.

To our knowledge, there has been no investigation into this direction yet. Although, for the smallest interesting values of k and r, a result of S. Guo and Z.-W. Sun [4] can be interpreted this way. Namely, if $k = s+1$, then the existence of $w(\mathcal{R}_{a_1,\dots,a_s}, k, 2)$ follows from a less-known theorem of R. Rado [11], since the linear equation $a_1x_1 + \cdots + a_sx_s - x_{s+1} = 0$ with three or more unknowns has both positive and negative coefficients. The exact value of this van der Waerden type number is $w(\mathcal{R}_{a_1,\dots,a_s}, k, 2) = ab^2 + b - a$, where $a = \min\{a_1, \dots, a_s\}$ and $b = a_1 + \cdots + a_s$.

We prove our first result using Rado's Theorem.

Theorem 2.1. *Let $s \geq 2$ and $a_1, \dots, a_s$ be positive integers.*

1. *For $k = s+1$, $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ exists for all $r \geq 1$ if and only if $a_1 = 1$ or $\dots$ or $a_s = 1$.*
2. *For $k = s+2$, $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ exists for all $r \geq 1$ if and only if $a_1 = a_s = 1$.*
3. *For $k \geq s+3$, $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ does not exist for all $r \geq 1$.*

Proof.

1. If $k = s+1$, then in fact we look for a monochromatic positive integer solution of $a_1x_1 + \cdots + a_sx_s - x_{s+1} = 0$.
   The sum of some coefficients can be 0 if and only if $a_i = 1$ for some $1 \leq i \leq s$, hence applying Rado's Theorem we conclude that this equation is regular precisely in this case.
2. If $k = s+2$, then we have to check the regularity of the homogeneous system of linear equations with coefficient matrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{s-1} & a_s & -1 & 0 \\ 0 & a_1 & a_2 & \dots & a_{s-1} & a_s & -1 \end{pmatrix}.$$

   Denote by c_i the ith column of this matrix $(i = 1, \dots, k)$. We verify the columns condition for it.

   If there exist columns with zero vector sum, then the sum of all of their coordinates is 0, as well. The first, the $(s+1)$st and the $(s+2)$nd column sums are $a_1 \geq 1$, $a_s - 1 \geq 0$ and -1, respectively, while the ith column sum is $a_i + a_{i-1} \geq 2$ $(i = 2, \dots, s)$. The sum of these column sums can be 0 only when we choose the columns
   – c_{s+1} and $a_s = 1$,
   – c_1, c_{s+2}, and $a_1 = 1$,
   – c_{s+1}, c_{s+2}, and $a_s = 2$,
   – c_1, c_{s+1}, c_{s+2}, and $a_1 = a_s = 1$.
   The sums of the chosen columns are

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   respectively. Hence only the last case remains, when the sum of the other columns (i.e. $c_2 + \cdots + c_s$) can be written as a linear combination of the chosen ones by

$$(a_2 + \cdots + a_s) \cdot c_1 + 0 \cdot c_{s+1} - (a_1 + \cdots + a_{s-1}) \cdot c_{s+2}.$$

   This means that the above system of linear equations is regular if and only if $a_1 = a_s = 1$ by Rado's Theorem.
3. If $k \geq s+3$, then the coefficient matrix of the homogeneous system of linear equations under investigation is

$$\begin{pmatrix}
a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \dots & \vdots \\
0 & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & 0 \\
0 & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & 0 \\
\vdots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1
\end{pmatrix}.$$

Denote its ith column by c_i $(i = 1, \dots, k)$. Similarly to the above case, we consider again the column sums.

If $s \geq 3$, or if $s = 2$ and a_1, a_2 are not both 1, then the first, the $(k-2)$nd, the $(k-1)$st, and the kth column sums are $a_1 \geq 1$, $a_{s-1} + a_s - 1 \geq 1$, $a_s - 1 \geq 0$, and -1, respectively. Finally, the ith column sum is at least 2 for $2 \leq i \leq k-3$. The sum of these column sums can be 0 only in those cases when we choose columns

– c_{k-1} and $a_s = 1$,
– c_1, c_k, and $a_1 = 1$,
– c_{k-2}, c_k, and $a_{s-1} = a_s = 1$,
– c_{k-1}, c_k, and $a_s = 2$,
– c_1, c_{k-1}, c_k, and $a_1 = a_s = 1$,
– c_{k-2}, c_{k-1}, c_k, and $a_{s-1} = a_s = 1$.

If $s = 2$ and $a_1 = a_2 = 1$, then the first, the second, the ith $(3 \leq i \leq k-2)$, the $(k-1)$st and the kth column sums are 1, 2, 1, 0 and -1, respectively. Now the sum of these column sums can be 0 only if we choose c_{k-1}, or one of $c_1, c_3, c_4, \dots, c_{k-2}$ and c_k, or one of $c_1, c_3, c_4, \dots, c_{k-2}$ and c_{k-1} and c_k.

But none of these cases is suitable, because the sum of the chosen columns is never the zero vector. Thus it follows from Rado's Theorem that our system of linear equations is not regular. $\square$

In those cases, when $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ does not exist for all $r \geq 1$ by the above theorem, we give a concrete number of colors for which this van der Waerden type number does not exist.

Theorem 2.2. *Let $s \geq 2$ and $a_1, \dots, a_s$ be positive integers. Further, let p be the smallest prime such that $a_1 + \dots + a_s + 1 \leq p$.*

1. *Suppose that $a_1, \dots, a_s \geq 2$. If $k \geq s+1$ and $r \geq p-1$, then $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ does not exist.*
2. *Suppose that $a_1 \geq 2$ or $a_s \geq 2$. If $k \geq s+2$ and $r \geq p-1$, then $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ does not exist.*
3. *If $k \geq s+3$ and $r \geq p-1$, then $w(\mathcal{R}_{a_1,\dots,a_s}, k, r)$ does not exist.*

Proof. Throughout this proof consider the following $(p-1)$-coloring of positive integers: Every positive integer x can be uniquely written in the form $x = p^{\alpha}(p\beta + \gamma)$ where α, β are nonnegative integers and $\gamma \in \{1, \dots, p-1\}$. Then let the color of x be γ.

1. First we prove that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1,\dots,a_s}$ of length $s+1$, or in other words, a monochromatic solution of equation $a_1x_1 + \dots + a_sx_s = x_{s+1}$. Indirectly suppose that $x_i = p^{\alpha_i}(p\beta_i + \gamma)$ $(i = 1, \dots, s+1)$ are solutions of the equation with nonnegative integers α_i, β_i $(i = 1, \dots, s+1)$ and $\gamma \in \{1, \dots, p-1\}$, that

is

$$a_1p^{\alpha_1}(p\beta_1+\gamma)+\cdots+a_sp^{\alpha_s}(p\beta_s+\gamma)=p^{\alpha_{s+1}}(p\beta_{s+1}+\gamma).$$

Let $\alpha_{\min}=\min\{\alpha_1,\ldots,\alpha_s\}$ and $I=\{i\in\{1,\ldots,s\}\mid\alpha_i=\alpha_{\min}\}$. Since $p^{\alpha_{\min}}$ divides the left-hand side of the equation, it also divides the right-hand side, hence, $\alpha_{s+1}\geq\alpha_{\min}$.

After dividing by $p^{\alpha_{\min}}$ and reducing modulo p, we obtain $\sum\limits_{i\in I}a_i\gamma\equiv 0$ or $\gamma \pmod p$ depending on whether $\alpha_{s+1}>\alpha_{\min}$ or $\alpha_{s+1}=\alpha_{\min}$. By the assumptions, we know that $2\leq\sum\limits_{i\in I}a_i\leq p-1$, therefore the previous congruence is equivalent to $\gamma\equiv 0 \pmod p$, which is a contradiction.

2. Now we show that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1,\ldots,a_s}$ of length $s+2$, that is the system of equations

$$\begin{aligned}a_1x_1+a_2x_2+\cdots\ +a_sx_s&=x_{s+1}\\ a_1a_sx_1+(a_1+a_2a_s)x_2+\cdots\ +(a_{s-1}+a_s^2)x_s&=x_{s+2}\end{aligned}$$

has no monochromatic solution. Similarly to the proof of the first part, suppose that there exists a monochromatic solution and use the same notation. Then we have $\alpha_{s+1},\alpha_{s+2}\geq\alpha_{\min}$.

If $I=\{i\}$ and $a_i\geq 2$ $(1\leq i\leq s)$, or $|I|\geq 2$, then the statement follows the same way as in the first part.

If $I=\{1\}$ and $a_1=1$, then dividing the second equation by $p^{\alpha_{\min}}$ and reducing modulo p give $a_s\gamma\equiv 0$ or $\gamma \pmod p$. This leads similarly to a contradiction since $2\leq a_s\leq p-1$.

If $I=\{i\}$ $(2\leq i\leq s)$ and $a_i=1$, then by the above argument we obtain from the second equation that $(a_{i-1}+a_s)\gamma\equiv 0$ or $\gamma \pmod p$, which implies again a contradiction because $2\leq a_{i-1}+a_s\leq p-1$.

3. Finally, we prove that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1,\ldots,a_s}$ of length $s+3$, which means that the following system of equations has no monochromatic solution:

$$\begin{aligned}a_1x_1+a_2x_2+a_3x_3+\cdots\ +a_sx_s&=x_{s+1}\\ a_1a_sx_1+(a_1+a_2a_s)x_2+(a_2+a_3a_s)x_3+\cdots\ +(a_{s-1}+a_s^2)x_s&=x_{s+2}\\ b_1x_1+b_2x_2+b_3x_3+\cdots\ +b_sx_s&=x_{s+3}\end{aligned}$$

$$\text{(where } b_1=a_1a_{s-1}+a_1a_s^2,\ b_2=a_2a_{s-1}+a_1a_s+a_2a_s^2,$$
$$b_3=a_1+a_3a_{s-1}+a_2a_s+a_3a_s^2,\ldots,b_s=a_{s-2}+2a_{s-1}a_s+a_s^3).$$

Suppose that we have a monochromatic solution and use the same notation as in the previous parts. Then we have again $\alpha_{s+1},\alpha_{s+2},\alpha_{s+3}\geq\alpha_{\min}$.

If $I=\{1\}$ and $a_1\geq 2$, or $I=\{1\}$ and $a_s\geq 2$, or $I=\{i\}$ $(2\leq i\leq s)$, or $|I|\geq 2$, then the proof goes along the lines of the proof of the second part.

Moreover, if $I=\{1\}$ and $a_1=a_s=1$, then divide the third equation by $p^{\alpha_{\min}}$ and reduce it modulo p to get $(a_{s-1}+a_s)\gamma\equiv 0$ or $\gamma \pmod p$, which gives a contradiction since $2\leq a_{s-1}+a_s\leq p-1$.

□

Remark 2.3. *Apart from the cases $s = 2$, $a_1 = 1$, and $s = 2$, $a_2 = 1$, it could be shown that in the second and third parts of the theorem p can be chosen to be the smallest prime for which $a_1 + \cdots + a_s \le p$, because in the case of $a_1 + \cdots + a_s = p$ and $I = \{1, \ldots, s\}$ we can use the second equation instead of the first one.*

If k is large enough, then a similar theorem can be proved for an arbitrary number of colors.

Theorem 2.4. *Let $s \ge 2$ and $a_1, \ldots, a_s$ be positive integers. If $k \ge 2s + 1$ and $r \ge 2$, then $w(\mathcal{R}_{a_1,\ldots,a_s}, k, r)$ does not exist.*

Proof. For $j \ge 0$, let $I_j = [(a_1 + \cdots + a_s)^j, (a_1 + \cdots + a_s)^{j+1} - 1] \cap \mathbb{Z}$. Then $\{I_j \mid j \ge 0\}$ is a partition of the positive integers. Color the elements of I_j with 0 if j is even, and color them with 1 if j is odd.

Indirectly suppose that there exists a sequence $(x_i)_{i=1}^{\infty}$ of $\mathcal{R}_{a_1,\ldots,a_s}$ such that the first $2s+1$ elements are monochromatic.

Let $m \in \{1, \ldots, s\}$ be defined by $x_m = \max\{x_1, \ldots, x_s\}$, and t be the exponent such that

$$(a_1 + \cdots + a_s)^t \le x_m < (a_1 + \cdots + a_s)^{t+1}.$$

Then we can derive the inequalities

$$(a_1 + \cdots + a_s)^t \le x_m \le a_m x_m \le a_1 x_1 + \cdots + a_s x_s = x_{s+1},$$

$$x_{s+1} = a_1 x_1 + \cdots + a_s x_s \le (a_1 + \cdots + a_s) x_m < (a_1 + \cdots + a_s)^{t+2}.$$

Since x_m and x_{s+1} share their colors, it follows that $(a_1+\cdots+a_s)^t \le x_{s+1} < (a_1+\cdots+a_s)^{t+1}$. Now, from $x_2, \ldots, x_s \le x_m \le x_{s+1}$ we obtain the inequalities

$$(a_1 + \cdots + a_s)^t \le x_{s+1} \le a_s x_{s+1} \le a_1 x_2 + \cdots + a_s x_{s+1} = x_{s+2},$$

$$x_{s+2} = a_1 x_2 + \cdots + a_s x_{s+1} \le (a_1 + \cdots + a_s) x_{s+1} < (a_1 + \cdots + a_s)^{t+2}.$$

Since x_{s+1} and x_{s+2} have the same color, we get $(a_1 + \cdots + a_s)^t \le x_{s+2} < (a_1 + \cdots + a_s)^{t+1}$. Similarly, we can deduce that $x_{s+1} \le x_{s+2} \le \cdots \le x_{2s}$ and $(a_1 + \cdots + a_s)^t \le x_j < (a_1 + \cdots + a_s)^{t+1}$ $(j = s+1, \ldots, 2s)$.

Finally, the inequalities

$$(a_1 + \cdots + a_s)^{t+1} \le (a_1 + \cdots + a_s) x_{s+1} \le a_1 x_{s+1} + \cdots + a_s x_{2s} = x_{2s+1},$$

$$x_{2s+1} = a_1 x_{s+1} + \cdots + a_s x_{2s} \le (a_1 + \cdots + a_s) x_{2s} < (a_1 + \cdots + a_s)^{t+2}$$

hold, hence, $(a_1 + \cdots + a_s)^{t+1} \le x_{2s+1} < (a_1 + \cdots + a_s)^{t+2}$, which means that the color of x_{2s+1} differs from the color of the previous elements. □

Remark 2.5. *Denote by $\widetilde{\mathcal{R}}_{a_1,\ldots,a_s}$ the set of those sequences $(x_i)_{i=1}^{\infty}$ of $\mathcal{R}_{a_1,\ldots,a_s}$ for which $x_s = \max\{x_1, \ldots, x_s\}$. For this family, a stronger result can be proved by the above argument. Let $s \ge 2$ and $a_1, \ldots, a_s$ be positive integers. If $k \ge 2s$ and $r \ge 2$, then $w(\widetilde{\mathcal{R}}_{a_1,\ldots,a_s}, k, r)$ does not exist.*

3. Multibonacci Sequences

For $s \ge 2$, denote by $\mathcal{M}_s = \mathcal{R}_{1,\ldots,1}$ the family of multibonacci sequences of order s, that is the set of sequences $(x_i)_{i=1}^{\infty}$ containing positive integers and satisfying the linear recurrence $x_i = x_{i-s} + \cdots + x_{i-1}$ $(i \ge s+1)$. For example, $\mathcal{M}_2 = \mathcal{F}$.

We notice that for $k = s+1$, $w(\mathcal{M}_s, k, 2)$ exists and $w(\mathcal{M}_s, k, 2) = s^2 + s - 1$, which was proved by A. Beutelspacher and W. Brestovansky [3] before the general case in [4].

In addition, for $s \geq 2$, let $\widetilde{\mathcal{M}}_s = \widetilde{\mathcal{R}}_{1,\ldots,1}$, that is the family of sequences $(x_i)_{i=1}^{\infty}$ of $\mathcal{M}_s$ for which $x_s = \max\{x_1, \ldots, x_s\}$. Especially, $\widetilde{\mathcal{M}}_2 = \widetilde{\mathcal{F}}$.

Our Theorems 2.1, 2.2 (together with Remark 2.3), 2.4, and Remark 2.5 in the previous section give the following consequences for $\mathcal{M}_s$ and $\widetilde{\mathcal{M}}_s$.

Theorem 3.1.

1. *Let $s \geq 2$ and $k \geq s+1$. Then $w(\mathcal{M}_s, k, r)$ exists for all $r \geq 1$ if and only if $k = s+1$ or $s+2$.*
2. *Let $s \geq 3$ and p be the smallest prime for which $s \leq p$. If $k \geq s+3$ and $r \geq p-1$, then $w(\mathcal{M}_s, k, r)$ does not exist.*
3. *Let $s \geq 2$. If $k \geq 2s+1$ and $r \geq 2$, then $w(\mathcal{M}_s, k, r)$ does not exist.*
4. *Let $s \geq 2$. If $k \geq 2s$ and $r \geq 2$, then $w(\widetilde{\mathcal{M}}_s, k, r)$ does not exist.*

In our final remark, we make a few observations about the coloring used in the proof of Theorem 2.2 through multibonacci sequences.

Remark 3.2. *Following the lines of the proof of Theorem 2.2, for $\mathcal{M}_s$ the system of equations is*

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_s &= x_{s+1} \\ x_1 + 2x_2 + 2x_3 + \cdots + 2x_s &= x_{s+2} \\ 2x_1 + 3x_2 + 4x_3 + \cdots + 4x_s &= x_{s+3}. \end{aligned}$$

The second part of Theorem 3.1 is the best possible which can be achieved by this coloring in the following sense. We cannot choose a smaller prime p, since in the case of $s = p+1$ and $\alpha_1 = \cdots = \alpha_s$ we would obtain $(p+1)\gamma \equiv 0$ or $\gamma \pmod p$, $(2p+1)\gamma \equiv 0$ or $\gamma \pmod p$, $(4p+1)\gamma \equiv 0$ or $\gamma \pmod p$ from the above equations, respectively, but these give no contradiction.

From a different point of view, we have used all three equations to obtain the second part of Theorem 3.1. Now suppose that $s \geq 4$. If $I = \{2\}$, or $I = \{i\}$ $(3 \leq i \leq s)$, or $s = p$ and $I = \{1, \ldots, s\}$, then we could use the third equation instead of the second one to obtain $3\gamma \equiv 0$ or $\gamma \pmod p$, $4\gamma \equiv 0$ or $\gamma \pmod p$, $(4p-3)\gamma \equiv 0$ or $\gamma \pmod p$, respectively. All of these cases result in a contradiction, since $p \geq 5$ when $s \geq 4$. It means that for $s \geq 4$, the second equation can be omitted, hence $x_1, \ldots, x_s, x_{s+1}, x_{s+3}$ cannot be monochromatic under this coloring.

In a similar way, for $s = 2$, with the choice of $p = 3$, we can show that if $k \geq 5$ and $r \geq 2$, then $w(\mathcal{M}_2, k, r)$ does not exist. Moreover, we can omit the first equation, and x_1, x_2, x_4, x_5 cannot be monochromatic.

References

[1] H. Ardal, Z. Dvořák, V. Jungić and T. Kaiser, *On a Rado type problem for homogeneous second order linear recurrences*, Electron. J. Combin., **17** (2010), #R38.

[2] H. Ardal, D. S. Gunderson, V. Jungić, B. M. Landman and K. Williamson, *Ramsey results involving the Fibonacci numbers*, The Fibonacci Quarterly, **46/47.1** (2008/2009), 10–17.

[3] A. Beutelspacher and W. Brestovansky, *Generalized Schur numbers*, Combinatorial Theory (eds. D. Jungnickel and K. Vedder), Lecture Notes in Math., Vol. 969, Springer, 1982, 30–38.

[4] S. Guo and Z.-W. Sun, *Determination of the two-color Rado number for $a_1x_1 + \cdots + a_mx_m = x_0$*, J. Combin. Theory, Ser. A, **115** (2008), 345–353.

[5] H. Harborth and S. Maasberg, *Rado numbers for homogeneous second order linear recurrences – degree of partition regularity*, Congr. Numer., **108** (1995), 109–118.

[6] H. Harborth and S. Maasberg, *Rado numbers for Fibonacci sequences and a problem of S. Rabinowitz*, Applications of Fibonacci Numbers (eds. G. E. Bergum, A. N. Philippou and A. F. Horadam), Vol. 6, Kluwer Acad. Publ., 1996, 143–153.
[7] H. Harborth and S. Maasberg, *Rado numbers for* $a(x+y)=bz$, J. Combin. Theory, Ser. A, **80** (1997), 356–363.
[8] B. M. Landman, *Ramsey functions associated with second order recurrences*, J. Combin. Math. Combin. Comput., **15** (1994), 119–127.
[9] B. M. Landman and A. Robertson, *Ramsey Theory on the Integers*, American Math. Soc., 2004.
[10] G. Nyul and B. Rauf, *Upper bounds on van der Waerden type numbers for some second order linear recurrence sequences*, Annales Math. Inform., **38** (2011), 117–122.
[11] R. Rado, *Studien zur Kombinatorik*, Math. Zeitschrift, **36** (1933), 424–480.
[12] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jahresber. Deutsch. Math.-Verein., **25** (1916), 114–117.
[13] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde, **15** (1927), 212–216.

MSC2010: 05D10, 11B37

Institute of Mathematics, University of Debrecen, H–4010 Debrecen P.O.Box 12, Hungary
E-mail address: `gnyul@science.unideb.hu`

Institute of Mathematics, University of Debrecen, H–4010 Debrecen P.O.Box 12, Hungary
E-mail address: `raufb@science.unideb.hu`