GOLDEN AND SILVER RATIOS IN BARGAINING

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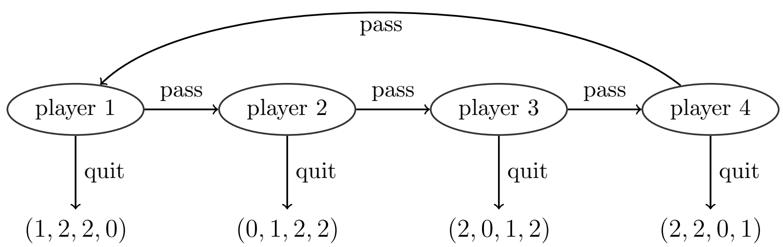
ABSTRACT. We examine a specific class of bargaining problems where the golden and silver ratios appear in a natural way.

1. INTRODUCTION

There are five coins on a table to be divided among players 1 to 4. The players act in turns cyclically starting from player 1. When it is a player's turn, he has two options: to quit or to pass. If he quits he earns a coin, the next two players get two coins each and the remaining player gets nothing. For example, if player 3 quits then he gets one coin, players 4 and 1 get two coins each and player 2 gets no coins. This way the game is repeated until someone quits. If nobody ever quits then they all get nothing. Each player can randomize between the two alternatives and maximizes his expected payoff. The game is presented in the figure below, where the players' payoffs are shown as the components of the vectors.



Bargaining problems have been studied extensively in game theory literature [4, 6, 7]. Our game can be seen as a bargaining problem where the players cannot make their own offers but can only accept or decline the given division. Furthermore, the game is a special case of a so-called sequential quitting game which has been studied, for example, in [5, 9]. The prominent solution concept in such games is a Nash equilibrium, which defines the players' moves such that no player can gain by any unilateral deviation. Moreover, we assume credible behavior throughout the play. Such "subgame-perfect" solutions exist by [5].

While studying structural properties of the subgame-perfect equilibria of the above game, we noticed that the golden ratio plays an important role in the strategies as well as in the distribution of payoffs. In particular, player 3 will get a corresponding payoff of $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, player 2 will get $3 - \varphi$ and players 1 and 4 will get 1, i.e. the golden ratio is used to split the additional fifth coin in probability amongst players 2 and 3. In further explorations, we observed that for several other games of this type, not only the golden but also the silver ratio appears in the solutions. A common feature of these games is that n players have to split $n + 1$ coins according to some specific rules. In this respect, we also relate to the earlier literature for dividing the dollar [1, 2].

Kimmo Berg acknowledges funding from Emil Aaltosen Säätiö through Post doc -pooli.

The games examined here are closely related to quitting games in which the players all move simultaneously. For that class of games, the existence of (approximate) equilibria is one of the most challenging open problems in the field of dynamic games (cf. [3, 8, 10, 11]).

2. The Role of the Golden Ratio in the Solutions

In this section, we derive the equilibria for the above game and find that the golden ratio appears in the solution. Moreover, we extend the four-player game with cyclically symmetric payoff structure[1] $(1, 2, 2, 0)$ to an n-player game with payoff structure $(1, 1, \ldots, 1, 2, 2, 0)$.

In these games, a player's strategy defines the probability of quitting for each turn he has. In general, a player could assign different probabilities for the first turn, the second turn and so on. However, we restrict our analysis to stationary strategies, where for each player the probability that he uses is the same in all turns. A Nash equilibrium specifies a strategy for each player such that it is a best response to the other players' strategies. For example, if the player quits with probability one, then it must be that the quitting payoff is at least as high as the expected continuation payoff when passing, given the other players' strategies. Subgame-perfection requires that the same strategies must also form a Nash equilibrium when any of the other players start the game.

Theorem 2.1. *In the game with cyclically symmetric payoff structure* $(1, 2, 2, 0)$*, there are only two stationary subgame-perfect equilibria and both of them are symmetric in the players' strategies:*

(1) *The strategy profile in which every player quits with probability 1. The corresponding payoffs are* $(1, 2, 2, 0)$.

(2) *The strategy profile in which every player quits with probability* $p = \frac{3-\sqrt{5}}{2} = 2 - \varphi$. *According to this strategy profile, the overall probability that a certain player will eventually quit is given by the four-vector*

$$\tfrac{1}{3\varphi-4} \cdot (2-\varphi, -3+2\varphi, 5-3\varphi, -8+5\varphi)\,.$$

The corresponding payoffs are $(1, 3-\varphi, \varphi, 1)$.

Remark. Notice how the Fibonacci numbers appear in the total quitting probabilities mentioned in (2): 2, 3, 5, 8, and 1, 2, 3, 5 as the multipliers of φ. In fact, the probability that the game will end at stage $1, 2, 3, 4, \ldots$ is given respectively by: $2-\varphi, -3+2\varphi, 5-3\varphi, -8+5\varphi, 13-8\varphi, -21+13\varphi, 34-21\varphi, \ldots$.

Proof. Consider an arbitrary stationary subgame-perfect equilibrium and denote by p_i the probability that player $i \in \{1, 2, 3, 4\}$ puts on quitting in each of his turns. We will only determine these quitting probabilities p_i, then the expressions for the corresponding payoffs follow easily. We distinguish three cases.

Case 1. A player quits with probability 1, i.e., $p_i = 1$ for some player $i \in \{1, 2, 3, 4\}$.

Due to the cyclic symmetry, we may assume without loss of generality that $p_1 = 1$. Consequently, it is the best response for player 4 to quit, thus $p_4 = 1$. Similarly, we obtain $p_3 = 1$ and $p_2 = 1$. Since it is also player 1's best response to quit, this results in a stationary subgame-perfect equilibrium, in which every player quits with probability 1.

[1] A cyclically symmetric payoff structure (a, b, c, d) is to be interpreted as given by the payoffs that players get upon quitting by player 1; players get (d, a, b, c) if player 2 quits, etc. This means that the quitting player always gets a, the next one b, etc.

Case 2. A player quits with probability 0, i.e., $p_i = 0$ for some player $i \in \{1, 2, 3, 4\}$.

We distinguish two subcases: (i) $p_i = 0$ for all $i \in \{1, 2, 3, 4\}$, (ii) $p_i = 0$ and $p_{i+1} > 0$ for some i. Subcase (i) is not an equilibrium since any player should quit if nobody is quitting. In subcase (ii), the assumptions imply that player $i-1$ should pass, i.e., $p_{i-1} = 0$, as he then gets payoff 2. Now, this implies that $p_{i-2} = 0$, which means that only player $i+1$ is quitting and this contradicts that $p_i = 0$.

Case 3. The players randomize between the alternatives, i.e., $p_i \in (0, 1)$ for every player $i \in \{1, 2, 3, 4\}$.

In this case, every player $i \in \{1, 2, 3, 4\}$ is indifferent between quitting and continuing. Quitting yields payoff 1 for player i. However, if player i continues, then with probability $p_{i+1} + (1 - p_{i+1})p_{i+2} + (1 - p_{i+1})(1 - p_{i+2})p_{i+3}$, one of the other players quits before it is player i's turn again. Given that this happens, player i receives an expected payoff of

$$C_i \ := \ \frac{p_{i+1} \cdot 0 + (1 - p_{i+1})p_{i+2} \cdot 2 + (1 - p_{i+1})(1 - p_{i+2})p_{i+3} \cdot 2}{p_{i+1} + (1 - p_{i+1})p_{i+2} + (1 - p_{i+1})(1 - p_{i+2})p_{i+3}}.$$

In the expression above, players 5, 6, and 7 are identified with players 1, 2, and 3, respectively. Since player i is indifferent between quitting and continuing, we have $C_i = 1$ for every player $i \in \{1, 2, 3, 4\}$.

Since $C_1 = 1$, we obtain

$$p_4 = \frac{p_2 - p_3 + p_2 p_3}{(1 - p_2)(1 - p_3)}. \tag{2.1}$$

Substituting (2.1) into $C_2 = 1$ yields

$$p_1 = \frac{-p_2 + 2p_3 - 2p_2 p_3}{1 - 2p_2}. \tag{2.2}$$

Further substituting (2.1) and (2.2) into $C_3 = 1$, we derive

$$p_3 = \frac{p_2 + (p_2)^2}{3 - 5p_2 + 2(p_2)^2}, \tag{2.3}$$

and then finally $C_4 = 1$ gives $p_2 = \frac{3-\sqrt{5}}{2}$. After substitution, we find that $p_1 = p_2 = p_3 = p_4 = \frac{3-\sqrt{5}}{2}$, which yields a symmetric stationary subgame-perfect equilibrium. □

We now extend the above four-player game with payoffs $(1, 2, 2, 0)$ to an n-player game with payoffs $(1, 1, \ldots, 1, 2, 2, 0)$. This n-player game has a similar symmetric solution with corresponding payoffs $(1, 1, \ldots, 3-\varphi, \varphi, 1)$. The common quitting probability p for each player in the n-player game satisfies the equation

$$1 - (1 - p)^{n-1} = 0 \cdot p + 2 \cdot p(1 - p) + 2 \cdot p(1 - p)^2 + p(1 - p)^3 \sum_{i=0}^{n-5} (1 - p)^i.$$

Using that $p(1 - p)^3 \sum_{i=0}^{n-5} (1 - p)^i = (1 - p)^3(1 - (1 - p)^{n-4})$, this simplifies to

$$1 - (1 - p)^3 = 0 \cdot p + 2 \cdot p(1 - p) + 2 \cdot p(1 - p)^2,$$

which is exactly the same equation that determines the solution to the four-player game.

In the next section, we will see more games which have the golden ratio in the solution.

3. Different Payoffs and the Silver Ratio

In this section, we examine all possibilities of dividing the five coins among the four players. The symmetric stationary subgame-perfect equilibria can be computed in the same way as in the previous section and are presented in Table 1. In the first two lines, the stars $*$ mean that the payoffs can be anything as long as they are non-negative integers and sum up to five. The payoff structure $(0, *, *, *)$ applies to $(0, 1, 2, 2)$ and $(0, 3, 2, 0)$ for example.

Table 1. The quitting probabilities and the corresponding payoffs for symmetric stationary subgame-perfect equilibria in different games.

game	quitting probability	corresponding payoffs
$(0, *, *, *)$	$p = 0$	$(0, 0, 0, 0)$
$(y \geq x, *, *, x)$	$p = 1$	$(y, *, *, x)$
$(1, 0, 4, 0)$	$p = \varphi - 1 \approx 0.62$	$(1, 2 - \varphi, 1 + \varphi, 1) \approx (1, 0.38, 2.62, 1)$
$(1, 1, 3, 0)$	$p = 1/2$	$(1, 1, 2, 1)$
$(1, 2, 2, 0)$	$p = 1 - 1/\varphi \approx 0.38$	$(1, 3 - \varphi, \varphi, 1) \approx (1, 1.38, 1.62, 1)$
$(1, 3, 1, 0)$	$p = 1 - 1/\sqrt{2} \approx 0.29$	$(1, 3 - \sqrt{2}, \sqrt{2}, 1) \approx (1, 1.59, 1.41, 1)$
$(1, 4, 0, 0)^a$	$p = 1 - 1/\alpha \approx 0.23$	$(1, 3 - \alpha, \alpha, 1) \approx (1, 1.70, 1.30, 1)$
$(2, 0, 0, 3)$	$p = (3 - \sqrt{3})/2 \approx 0.63$	$(2, \sqrt{3} - 1, 2 - \sqrt{3}, 2) \approx (2, 0.73, 0.27, 2)$

$(1, 0, 2, 2)$ has no symmetric stationary subgame-perfect equilibrium
$(1, y \leq 3, x \leq 1, *)$ has no symmetric stationary subgame-perfect equilibrium
[a] Here $\alpha = (\sqrt{13} - 1)/2$.

We observe that the two games with the golden ratio, $(1, 2, 2, 0)$ and $(1, 0, 4, 0)$, are not the only ones that give a special way to divide the extra coin. All the expected payoffs of player 3, i.e., $1 + \varphi$, 2, φ, $\sqrt{2}$ and $(\sqrt{13} - 1)/2$, are related to the silver ratio or the silver means. In fact, the golden ratio φ is also the first silver mean. The number $\sqrt{2}$ is the second silver mean minus one and the number $(\sqrt{13} - 1)/2$ is the third silver mean minus two. The number 2 is related to the zeroth silver mean. Thus, player 3 receives some special fraction in each case.

The game with payoffs $(1, 0, 2, 2)$ has no stationary subgame-perfect equilibrium, but it does have a cyclic subgame-perfect equilibrium where the players quit in the order $1, 4, 3, 2, \ldots$ with common probability $1 - 1/\varphi$. The corresponding payoffs are $(1, 3 - \varphi, \varphi, 1)$.

Note that the games $(1, y \leq 3, x \leq 1, *)$ have non-symmetric stationary equilibria, where either $p_1 = p_3 = 1$ and $p_2 = p_4 = 0$, or $p_1 = p_3 = 0$ and $p_2 = p_4 = 1$.

We remark that all the four-player games with payoffs $(1, x, 4 - x, 0)$ can be extended to n-player games with payoffs $(1, 1, \ldots, 1, x, 4 - x, 0)$, while the equilibrium probabilities and corresponding payoffs preserve the golden and silver ratios like in the example at the end of Section 2.

4. Concluding Remarks

In the previous sections we have demonstrated how some of the golden ratio solutions can be extended to games with more than four players. Conversely, there is no straightforward way to find which n-player games of this type have solutions based on the golden ratio. The reason is that any such analysis involves solving higher order polynomial equations.

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MSC2010: 11B39, 91A20, 91B26

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