A CONDITION FOR ANTI-DIAGONALS PRODUCT INVARIANCE ACROSS POWERS OF 2×2 MATRIX SETS CHARACTERIZING A PARTICULAR CLASS OF POLYNOMIAL FAMILIES

PETER J. LARCOMBE AND ERIC J. FENNESSEY

Abstract. Motivated by some recent work on a particular class of polynomial families associated with certain types of integer sequences, we formulate a sufficient condition under which the anti-diagonals products across sets of characterizing 2×2 matrices remain invariant as matrix power increases. Two proofs are given along with some examples.

1. Introduction

1.1. **A Theorem.** We begin with the main result.

Theorem 1.1. *Let*

$$\mathbf{M}(x) = \mathbf{M}(A(x), B(x), C(x)) = \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix} \tag{1.1}$$

be a 2×2 matrix with $A(x), B(x), C(x) \in \mathbb{Z}[x]$. Likewise, let $\mathbf{M}'(x) = \mathbf{M}(A'(x), B'(x), C'(x))$ be a matrix with the same structure. Then, for all $n \geq 1$, if $B(x) = B'(x)$ and $A(x)C(x) = A'(x)C'(x)$ the products of the anti-diagonals of $\mathbf{M}^n$ and $\mathbf{M}'^n$ are pairwise equal.

1.2. **Examples and Background.** Before we set out our proofs, we give a couple of simple numeric examples (where matrix elements taken from $\mathbb{Z}[x]$ are merely integers (order zero polynomials)) and a short background to Theorem 1.1.

Example 1.2. *Let*

$$\mathbf{M}_1(x) = \begin{pmatrix} -2 & 4 \\ -5 & 0 \end{pmatrix}, \qquad \mathbf{M}_2(x) = \begin{pmatrix} -2 & -2 \\ 10 & 0 \end{pmatrix}. \tag{1.2}$$

As $n = 1, 2, 3, 4, \ldots, 10, \ldots,$ we find that the anti-diagonals products of both matrices $\mathbf{M}^n_{1,2}(x)$ have the common values $-20, -80, -5120, -103680, \ldots, -6471396884480, \ldots$.

Example 1.3. *Let*

$$\mathbf{M}_1(x) = \begin{pmatrix} 1 & 2 \\ 6 & 0 \end{pmatrix}, \qquad \mathbf{M}_2(x) = \begin{pmatrix} 1 & -3 \\ -4 & 0 \end{pmatrix}, \qquad \mathbf{M}_3(x) = \begin{pmatrix} 1 & 12 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

As $n = 1, 2, 3, 4, \ldots, 10, \ldots,$ we find that the anti-diagonals products of all three matrices $\mathbf{M}^n_{1,2,3}(x)$ have the common values $12, 12, 2028, 7500, \ldots, 239795187852, \ldots$.

The above examples are straightforward in nature, and in the case where $A(x), B(x), C(x)$ are drawn from $\mathbb{Z}[x]$ a matrix such as $\mathbf{M}(x)$ is said to characterize a family of polynomials $\in \mathbb{Z}[x]$; specifically, a polynomial family $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots,$ is defined in general terms as

$$\alpha_n(x) = \alpha_n(A(x), B(x), C(x)) = (1, 0)\mathbf{M}^n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0, \tag{1.4}$$

with closed form (see, for instance, [2, Eq. (1.4), p. 349])

$$\alpha_n(x) = \frac{1}{2^{n+1}} \frac{[-B(x)+\rho(x)]^{n+1} - [-B(x)-\rho(x)]^{n+1}}{\rho(x)}, \tag{1.5}$$

where $\rho(x) = \rho(A(x), B(x), C(x)) = \sqrt{B^2(x) - 4A(x)C(x)}$.

Regarding a context for consideration of the polynomial family class described, the arguments $A(x), B(x), C(x)$ are, conventionally, those arising as (functional) coefficients of a governing quadratic equation

$$A(x)T^2(x) + B(x)T(x) + C(x) = 0 \tag{1.6}$$

for the (ordinary) generating function $T(x)$, of a particular integer sequence with which a namesake polynomial family $\alpha_n(A(x), B(x), C(x))$ is associated. There are a great number of such sequences and polynomial families, to say the least. Previous work has detailed results on the specific instances of Catalan, (Large) Schröder and Motzkin polynomials (see the references in [1]). It is a particular recent paper [2] on the general class of polynomial families which motivates this one.

We now give our two proofs of Theorem 1.1, in which $A(x), B(x), C(x)$ are assumed nonzero, followed by a concluding example.

2. The Proofs

Proof I. Our first proof relies on a previous result which has been used elsewhere and permits a very direct argument.

Proof. Any matrix of the form $\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}$ has an nth power which is expressible in terms of so called Catalan polynomials thus:

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n = \begin{pmatrix} P_n(-xy) & xP_{n-1}(-xy) \\ yP_{n-1}(-xy) & xyP_{n-2}(-xy) \end{pmatrix}. \tag{I.1}$$

Equation (I.1) can be seen in [2, Eq. (2.5), p. 351] where it has been deployed in a sufficiency argument for cross-family polynomial equality. Catalan polynomials are a particular family of polynomials characterized by the matrix instance $\mathbf{M}(x, -1, 1)$, the initial ones being $P_0(x) = P_1(x) = 1$, $P_2(x) = 1 - x$, $P_3(x) = 1 - 2x$, $P_4(x) = 1 - 3x + x^2$, $P_5(x) = 1 - 4x + 3x^2$, $P_6(x) = 1 - 5x + 6x^2 - x^3$, $P_7(x) = 1 - 6x + 10x^2 - 4x^3$, and so on. Writing

$$\mathbf{M}(x) = -B(x) \begin{pmatrix} 1 & -A(x)/B(x) \\ C(x)/B(x) & 0 \end{pmatrix} \tag{I.2}$$

and using (I.1) with $x = -A(x)/B(x)$, $y = C(x)/B(x)$, the anti-diagonals product of $\mathbf{M}^n(x)$ is immediate as $-A(x)B^{2n-2}(x)C(x)P^2_{n-1}(A(x)C(x)/B^2(x))$. It follows, therefore, that the anti-diagonals product of $\mathbf{M}'^n(x)$ is $-A'(x)B'^{2n-2}(x)C'(x)\ P^2_{n-1}(A'(x)C'(x)/B'^2(x))$, and so in turn that these two products are equal when $B(x) = B'(x)$ and $A(x)C(x) = A'(x)C'(x)$. □

We can, by way of example, easily check by hand the predicted anti-diagonals product of $\mathbf{M}^n(x)$ (formulated in terms of the nth Catalan polynomial $P_{n-1}(x)$) for a low order case. After some simple algebra we find, for $n = 4$,

$$\mathbf{M}^4(x) = \begin{pmatrix} -3A(x)B^2(x)C(x) + A^2(x)C^2(x) + B^4(x) & -A(x)B^3(x) + 2A^2(x)B(x)C(x) \\ -2A(x)B(x)C^2(x) + B^3(x)C(x) & -A(x)B^2(x)C(x) + A^2(x)C^2(x) \end{pmatrix}, \tag{2.1}$$

with anti-diagonals product

$$\begin{aligned}
&[-A(x)B^3(x) + 2A^2(x)B(x)C(x)][-2A(x)B(x)C^2(x) + B^3(x)C(x)] \\
&\qquad = -A(x)B^6(x)C(x) + 4A^2(x)B^4(x)C^2(x) - 4A^3(x)B^2(x)C^3(x) \\
&\qquad = -A(x)B^6(x)C(x)\left(1 - \frac{4A(x)C(x)}{B^2(x)} + \frac{4A^2(x)C^2(x)}{B^4(x)}\right) \\
&\qquad = -A(x)B^6(x)C(x)\left(1 - \frac{2A(x)C(x)}{B^2(x)}\right)^2 \\
&\qquad = -A(x)B^6(x)C(x)P_3^2(A(x)C(x)/B^2(x)). \qquad (2.2)
\end{aligned}$$

Many other cases, for higher values of n, have been verified algebraically by extensive computation.

Proof II. Our second proof is more of a first principles one, driven by a neat matrix diagonalization.

Proof. Let

$$\mathbf{N}(\alpha^*, \beta^*; z) = \begin{pmatrix} \alpha^* & \beta^*/z \\ z & 0 \end{pmatrix}, \qquad (\text{II}.1)$$

so that the general polynomial family characterizing matrix $\mathbf{M}(A(x), B(x), C(x))$ has an equivalent version in $\mathbf{N}(\alpha^*, \beta^*; z)$ through the simple relation

$$\mathbf{N}(-B(x), -A(x)C(x); -C(x)) = \mathbf{M}(A(x), B(x), C(x)). \qquad (\text{II}.2)$$

In view of (II.2), therefore, it suffices merely to show that the anti-diagonals product of the matrix $\mathbf{N}^n(\alpha^*, \beta^*; z)$ is independent of z in order to establish Theorem 1.1.

As alluded to above, this is achieved by decomposing $\mathbf{N}(\alpha^*, \beta^*; z)$ as

$$\mathbf{N}(\alpha^*, \beta^*; z) = \mathbf{P}(\alpha^*, \beta^*; z)\mathbf{D}(\alpha^*, \beta^*)\mathbf{P}^{-1}(\alpha^*, \beta^*; z), \qquad (\text{II}.3)$$

where

$$\mathbf{P}(\alpha^*, \beta^*; z) = \begin{pmatrix} \frac{\alpha^*-K}{2z} & \frac{\alpha^*+K}{2z} \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{D}(\alpha^*, \beta^*) = \begin{pmatrix} \frac{\alpha^*-K}{2} & 0 \\ 0 & \frac{\alpha^*+K}{2} \end{pmatrix}, \qquad (\text{II}.4)$$

and $K = K(\alpha^*, \beta^*) = \sqrt{\alpha^{*2} + 4\beta^*}$. Thus,

$$\begin{aligned}
\mathbf{N}^n(\alpha^*, \beta^*; z) &= [\mathbf{P}(\alpha^*, \beta^*; z)\mathbf{D}(\alpha^*, \beta^*)\mathbf{P}^{-1}(\alpha^*, \beta^*; z)]^n \\
&= \mathbf{P}(\alpha^*, \beta^*; z)\mathbf{D}^n(\alpha^*, \beta^*)\mathbf{P}^{-1}(\alpha^*, \beta^*; z). \qquad (\text{II}.5)
\end{aligned}$$

Raising $\mathbf{D}(\alpha^*, \beta^*)$ to the power n is trivial and, writing for convenience $S_n = S_n(\alpha^*, \beta^*) = [(\alpha^* + K)/2]^n$, $T_n = T_n(\alpha^*, \beta^*) = [(\alpha^* - K)/2]^n$, we find, after some non-trivial algebra,

$$\mathbf{N}^n(\alpha^*, \beta^*; z) = \begin{pmatrix} \frac{S_n+T_n+\alpha^*(S_n-T_n)/K}{2} & \frac{\beta^*(S_n-T_n)}{Kz} \\ \frac{(S_n-T_n)z}{K} & \frac{S_n+T_n-\alpha^*(S_n-T_n)/K}{2} \end{pmatrix}, \qquad (\text{II}.6)$$

whose anti-diagonals product $\beta^*(S_n - T_n)^2/K^2$ is indeed not dependent on z. Thus, we have our required result. □

Remark 2.1. Rewriting the top left-hand entry of $\mathbf{N}^n(\alpha^*, \beta^*; z)$ in (II.6) as $[(\alpha^* + K)^{n+1} - (\alpha^* - K)^{n+1}]/2^{n+1}K$, then on setting $\alpha^* = -B(x)$, $\beta^* = -A(x)C(x)$, $z = -C(x)$ and noting that $K(x) = K(\alpha^*(x), \beta^*(x)) = \sqrt{B^2(x) - 4A(x)C(x)} = \rho(x)$, the expression reduces precisely to $\alpha_n(x)$ (1.5), as of course it should in view of (1.4) and (II.2); this is a pleasing check on the form of $\mathbf{N}^n(\alpha^*, \beta^*; z)$.

Finally, although our context for the work here and elsewhere has been based on the notion that $A(x), B(x), C(x)$, and so $\alpha_n(x)$, lie in $\mathbb{Z}[x]$, we finish with a final example in which they are taken from $\mathbb{C}$ and Theorem 1.1 still holds (not surprisingly); with

$$\mathbf{M}_1(x) = \begin{pmatrix} -5+2i & -(7+19i) \\ 2(1-2i) & 0 \end{pmatrix}, \qquad \mathbf{M}_2(x) = \begin{pmatrix} -5+2i & 10(1-i) \\ -(4+5i) & 0 \end{pmatrix},$$
$$\mathbf{M}_3(x) = \begin{pmatrix} -5+2i & -(3+i) \\ 2(14-3i) & 0 \end{pmatrix}, \qquad \mathbf{M}_4(x) = \begin{pmatrix} -5+2i & -1+9i \\ 10i & 0 \end{pmatrix}, \tag{2.3}$$

we find computationally that the anti-diagonals products of $\mathbf{M}^n_{1,2,3,4}(x)$ are invariant for each $n = 1, 2, 3, \ldots$.

3. Summary

In this short paper we have provided two proofs of an observation which, although set in the context of a polynomial family class, holds in a wider sense; the nature of each proof is quite different. Work on these polynomials associated with integer sequences in the way described will continue, and their properties explored.

We emphasize that the invariance condition given for anti-diagonals products across matrix sets as described in the paper is a sufficient one. The existence of any necessary condition(s) for the same result is not established here, indeed this would appear to be a very problematic question to address which is left as an open one for any interested reader—our attempt to make progress is detailed briefly in an Appendix for completeness.

In summary, we note that a thorough review of the appropriate literature reveals the possibility of a fundamental product, of the type presented here for 2×2 matrices, going unnoticed until now or at least being absent from it. This is somewhat surprising, and it may even prove to be the case that the result detailed is but a special case of a more general one that applies to powers of matrices of arbitrary order—this, too, remains an unresolved conjecture at the present time.

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Appendix

We demonstrate here the difficulty in formulating any necessary condition(s) alluded to in the Summary.

With reference to Proof I, a necessary argument for anti-diagonals product equality across powers of $\mathbf{M}^{n+1}(x)$ and $\mathbf{M}'^{n+1}(x)$ would, for $n \geq 0$, require by assumption that

$$A(x)B^{2n}(x)C(x)P_n^2(A(x)C(x)/B^2(x)) = A'(x)B'^{2n}(x)C'(x)P_n^2(A'(x)C'(x)/B'^2(x)), \tag{A.1}$$

from which constraints on $A(x), A'(x), B(x), B'(x), C(x), C'(x)$ are to be determined. In order to make any sense of (A.1) it is not unreasonable to seek a closed form for $[x^s]\{P_n^2(x)\}$, the coefficient of x^s within the square of the $(n+1)$th Catalan polynomial $P_n(x)$. Noting that

$P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-x)^i$ (see, for example, Theorem 2.2 of [2]), then

$$\begin{aligned} P_n^2(x) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-x)^i \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j}(-x)^j \\ &= \sum_{i,j=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}\binom{n-j}{j}(-x)^{i+j} \\ &= \sum_{i,j=0}^{\infty} \binom{n-i}{i}\binom{n-j}{j}(-x)^{i+j} \\ &= \sum_{s\geq 0} \left[\sum_{i+j=s} \binom{n-i}{i}\binom{n-j}{j} \right] (-x)^s, \end{aligned} \tag{A.2}$$

so that $[x^s]\{P_n^2(x)\} = (-1)^s F(s,n)$ where

$$F(s,n) = \sum_{i=0}^{s} \binom{n-i}{i}\binom{n-s+i}{s-i}, \tag{A.3}$$

with $s = 0, 1, \ldots, n$ (n (even) $= 0, 2, 4, 6, \ldots$) and $s = 0, 1, \ldots, n-1$ (n (odd) $= 1, 3, 5, 7, \ldots$). After some algebra, we can express $F(s,n)$ as the finite hypergeometric series

$$F(s,n) = \binom{n-s}{s} {}_4F_3 \left(\left. \begin{matrix} n+1-s, \frac{1}{2}-\frac{1}{2}n, -\frac{1}{2}n, -s \\ \frac{1}{2}n+\frac{1}{2}-s, \frac{1}{2}n+1-s, -n \end{matrix} \right| 1 \right), \tag{A.4}$$

but it cannot, however, be summed to a closed form which halts our line of enquiry (thanks to Professor Dr. Wolfram Koepf for attempting to do so using his customized software).

References

[1] P. J. Larcombe and E. J. Fennessey, *A non-linear identity for a particular class of polynomial families*, The Fibonacci Quarterly, **52.1** (2014), 75–79.

[2] P. J. Larcombe and E. J. Fennessey, *Conditions governing cross-family member equality in a particular class of polynomial families*, The Fibonacci Quarterly, **52.4** (2014), 349–356.

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School of Computing and Mathematics, University of Derby, Kedleston Road, Derby DE22 1GB, England, U.K.
E-mail address: p.j.larcombe@derby.ac.uk

BAE Systems Integrated System Technologies, Broad Oak, The Airport, Portsmouth PO3 5PQ, England, U.K.
E-mail address: Eric.Fennessey@baesystems.com