A GRAPH-THEORETIC ENCODING OF LUCAS SEQUENCES

JAMES ALEXANDER AND PAUL HEARDING

ABSTRACT. Some well-known results of Prodinger and Tichy are that the number of independent sets in the n-vertex path graph is F_{n+2}, and that the number of independent sets in the n-vertex cycle graph is L_n. We generalize these results by introducing new classes of graphs whose independent set structures encode the Lucas sequences of both the first and second kind. We then use this class of graphs to provide new combinatorial interpretations of the terms of Dickson polynomials of the first and second kind.

1. INTRODUCTION AND MAIN RESULTS

For any graph G, we call a set S of vertices of G an *independent set* if no two vertices of S are adjacent. We let $i(G)$ denote the total number of independent sets of G and, for each $t \in \mathbb{N}$, we let $i_t(G)$ denote the number of independent sets of G of size t; thus, $i(G) = \sum_{t \geq 0} i_t(G)$. The quantity $i(G)$ was first explicitly considered by Prodinger and Tichy in [5], who referred to it as the Fibonacci number of a graph. We present two of their main results as the following theorem. Here, P_n denotes the n-vertex path graph, C_n denotes the n-vertex cycle graph (where the 1-vertex cycle is taken to be a vertex with a loop, and the 2-vertex cycle is taken to be a single edge), and we adopt the common conventions $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$.

Theorem 1.1. *For any $n \in \mathbb{N}$,*

$$i(P_n) = F_{n+2}, \tag{1.1}$$

and

$$i(C_n) = L_n. \tag{1.2}$$

Our main result will be a generalization of Theorem 1.1. For this, we will define two new classes of graphs. Fix any $n, a, b \in \mathbb{N}$ with $a \geq b$. Create an n-vertex cycle with vertex set $\mathbb{Z}_n$; for each vertex of the cycle, create an a-vertex complete graph sharing with the cycle only this vertex. Then, for each $v \in \mathbb{Z}_n$, make vertex v adjacent to $a - b$ additional vertices of the complete graph containing vertex $v + 1$ (mod n), and denote this graph $C(n, a, b)$. For example, $C(6, 5, 3)$ is the graph:
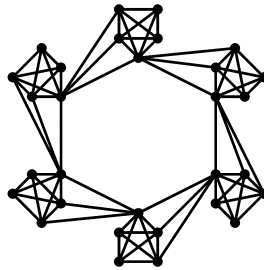
We refer to this class of graphs, over all valid $n, a, b \in \mathbb{N}$, as *chainsaw graphs*. When referring to a particular chainsaw graph $C(n, a, b)$, we call the n vertices lying on the inner cycle its

chain vertices, and we call the set of remaining vertices its *blade vertices*. This will serve as our generalization of C_n, as we will soon see. We generalize the path graph to a graph which we denote $P(n, a, b)$ by considering $C(n + 1, a, b)$ and removing one of the chain vertices (e.g., vertex 0) and all edges adjacent to it. We call these graphs *broken chainsaws*, and refer to the vertices similarly as chain and blade vertices.

With these definitions in place, we now state our generalization of Theorem 1.1. As is common, we let $U_n(a, b)$ and $V_n(a, b)$ denote the Lucas sequences of the first and second kind, respectively. That is, we let $U_0(a, b) = 0$, $U_1(a, b) = 1$, and $U_n(a, b) = aU_{n-1}(a, b) - bU_{n-2}(a, b)$ for $n > 1$ (so that $U_n(1, -1)$ are the Fibonacci numbers); we let $V_0(a, b) = 2$, $V_1(a, b) = a$, and $V_n(a, b) = aV_{n-1}(a, b) - bV_{n-2}(a, b)$ for $n > 1$ (so that $V_n(1, -1)$ are the Lucas numbers).

Theorem 1.2. *For any $n, a, b \in \mathbb{N}$ satisfying $a \geq b$, we have that*

$$i(P(n, a, b)) = U_{n+2}(a, -b), \tag{1.3}$$

and that

$$i(C(n, a, b)) = V_n(a, -b). \tag{1.4}$$

We prove this Theorem in Section 2 while discussing some relationships between Dickson polynomials and Lucas sequences and providing some graph-theoretic interpretations of these well-studied objects. We note that Theorem 1.1 is the special case of Theorem 1.2 when $a = b = 1$.

2. Relationships to Dickson Polynomials and a Proof of Theorem 1.2

In this section we examine the relationship between Dickson Polynomials and Lucas sequences and discuss some results which will be crucial to proving Theorem 1.2. In the process, we provide new graph-theoretic interpretations of Lucas sequences and Dickson polynomials. As is common, we use $D_n(X, Y)$ and $E_n(X, Y)$ to denote Dickson polynomials of the first and second kind, respectively. That is, we let

$$D_n(X, Y) := \sum_{t=0}^{\lfloor n/2 \rfloor} \frac{n}{n-t} \binom{n-t}{t} (-Y)^t X^{n-2t}, \tag{2.1}$$

and

$$E_n(X, Y) := \sum_{t=0}^{\lfloor n/2 \rfloor} \binom{n-t}{t} (-Y)^t X^{n-2t}. \tag{2.2}$$

We start with the following result which is known in finite field theory. See, for example, [2, 3], or [4] for more on this result. For more information on Dickson polynomials in general, see [4].

Theorem 2.1. *For any $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$,*

$$D_n(a, b) = V_n(a, -b), \tag{2.3}$$

and

$$E_n(a, b) = U_{n+1}(a, -b). \tag{2.4}$$

We will prove Theorem 1.2 by showing that the tth term of (2.1) and the tth term of (2.2) can be graph-theoretically interpreted as the number of independent sets in the chainsaw graph $C(n, a, b)$ and the broken chainsaw graph $P(n, a, b)$, respectively, which contain exactly t chain

vertices. For this, we will need the following result, which is well-known in graph theory, and is not difficult to prove. See, for example, [1].

Lemma 2.2. *For any $n \in \mathbb{N}$ and $t \in \mathbb{N}_0$, we have*

$$i_t(P_n) = \binom{n - t + 1}{t}, \tag{2.5}$$

and we also have

$$i_t(C_n) = \frac{n}{n - t}\binom{n - t}{t}. \tag{2.6}$$

With this in place, we are now ready to proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $n, a, b \in \mathbb{N}$ so that $a \geq b$. As previously discussed, it follows from (2.2) and (2.5) that (1.3) holds if the number of independent sets in $P(n, a, b)$ which contain exactly t chain vertices for $t \in \mathbb{N}_0$ is given by

$$i_t(P_n)b^t a^{n-2t+1}. \tag{2.7}$$

First, note that the number of ways to choose t independent chain vertices in $P(n, a, b)$, by definition, is $i_t(P_n)$. Then, once t independent chain vertices are chosen, there are t sets of $b-1$ blade vertices and $n - 2t + 1$ sets of $a - 1$ blade vertices with which they share no adjacencies, so (2.7) holds. A similar argument shows that the number of independent sets in $C(n, a, b)$ which contain exactly t chain vertices for $t \in \mathbb{N}_0$ is given by

$$i_t(C_n)b^t a^{n-2t}, \tag{2.8}$$

and thus, by (2.1) and (2.6), we have (1.4). $\square$

A graph-theoretic interpretation of the Lucas Sequence is now established by Theorem 1.2, and from its proof emerges a graph-theoretic interpretation of the terms of the Dickson polynomial.

3. Acknowledgments

References

[1] G. Hopkins and W. Staton, *Some identities arising from the Fibonacci numbers of certain graphs*, The Fibonacci Quarterly, **22.3** (1984), 255–258.

[2] C. Kimberling, *Divisibility properties of recurrent sequences*, The Fibonacci Quarterly, **14.4** (1976), 369–376.

[3] C. Kimberling, *Greatest common divisors of sums and differences of Fibonacci, Lucas, and Chebyshev polynomials*, The Fibonacci Quarterly, **17.1** (1979), 18–22.

[4] R. Lidl, G. L. Mullen, and R. Turnwald, *Dickson Polynomials*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 1965

[5] H. Prodinger and R. Tichy, *Fibonacci numbers of graphs*, The Fibonacci Quarterly, **20.1** (1982), 16–21.

MSC2010: 05C30, 05C38, 11Y55

Department of Mathematical Sciences, University of Delaware, Newark, Delaware
E-mail address: alex@math.udel.edu

Department of Mathematical Sciences, University of Delaware, Newark, Delaware
E-mail address: hearding@math.udel.edu