THE ZECKENDORF REPRESENTATION OF A BEATTY-RELATED FIBONACCI SUM

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Abstract. We obtain here the Zeckendorf representation of a sum of Fibonacci numbers indexed by a particular Beatty sequence known as the lower Wythoff sequence.

1. Introduction

Let $\lfloor x \rfloor$ be the *floor function*, denoting the largest integer not exceeding x, and let $\alpha > 1$ be an irrational number. The strictly increasing sequence of positive integers $\mathcal{B}(\alpha) = (\lfloor n\alpha \rfloor)_{n \geq 1}$ is known as a *Beatty sequence*. We are interested here in a particular Beatty sequence known as the *lower Wythoff sequence*. It is given by $\mathcal{B}(\phi)$, where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

is the *golden ratio*. In this paper we obtain the Zeckendorf representation for

$$S_n = \sum_{k=1}^{n} F_{\lfloor k\phi \rfloor}, \tag{1.1}$$

the sum of the Fibonacci numbers indexed by the first n terms of $\mathcal{B}(\phi)$. Some preliminary results are given in Section 2, and the main theorem is proved in Section 3.

2. Some Initial Results

Zeckendorf's Theorem [1, 3, 4] states that every $n \in \mathbb{N}$ has a unique representation as the sum of distinct Fibonacci numbers that does not include any consecutive Fibonacci numbers. Somewhat more formally, for any $n \in \mathbb{N}$ there exists an increasing sequence of positive integers of length $k \in \mathbb{N}$, $(c_1, c_2, \ldots, c_k)$ say, such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and

$$n = \sum_{i=1}^{k} F_{c_i}.$$

Relatively straightforward proofs of this result are given in [1, 3]. Note that the representation of S_n as a sum of Fibonacci numbers as given in (1.1) is not, in general, its Zeckendorf representation. Indeed, there exist values of k for which $\lfloor k\phi \rfloor$ and $\lfloor (k+1)\phi \rfloor$ are consecutive integers.

As we shall see, the complementary sequence to $\mathcal{B}(\phi)$ is also of relevance here. It is given by $\mathcal{B}\left(\phi^2\right) = \left(\lfloor n\phi^2 \rfloor\right)_{n \geq 1}$, and termed the *upper Wythoff sequence*. We will make use of the fact that, as a pair of complementary sequences, $\mathcal{B}(\phi)$ and $\mathcal{B}(\phi^2)$ satisfy both $\mathcal{B}(\phi) \cap \mathcal{B}(\phi^2) = \emptyset$ and $\mathcal{B}(\phi) \cup \mathcal{B}(\phi^2) = \mathbb{N}$.

The notation $\{x\}$ will be adopted to represent $x - \lfloor x \rfloor$, the *fractional part* of x. It is the case that $0 \leq \{x\} < 1$ for any $x \in \mathbb{R}$, but, for each $n \in \mathbb{N}$, the irrationality of ϕ implies that

$0 < \{n\phi\} < 1$. We will also make use of the equality $\phi^2 = \phi + 1$ and its many rearrangements throughout.

We now give a lemma concerning Beatty sequences, a sketch proof of which is given in [2]. For the sake of both clarity and completeness, however, we provide a detailed proof here. There then follow two further lemmas.

Lemma 2.1. *Let $\alpha > 1$ be an irrational number and j a positive integer. Then $j \in \mathcal{B}(\alpha)$ if and only if,*

$$0 < 1 - \frac{1}{\alpha} < \left\{\frac{j}{\alpha}\right\}.$$

Proof. First, we have

$$j = \left\lfloor \frac{j}{\alpha} \right\rfloor \alpha + \left\{\frac{j}{\alpha}\right\} \alpha \tag{2.1}$$

$$= \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha - \left(1 - \left\{\frac{j}{\alpha}\right\}\right) \alpha. \tag{2.2}$$

Then, since

$$\left\{\frac{j}{\alpha}\right\} \alpha > 0 \quad \text{and} \quad \left(1 - \left\{\frac{j}{\alpha}\right\}\right) \alpha > 0,$$

it follows from (2.1) and (2.2) that

$$\left\lfloor \frac{j}{\alpha} \right\rfloor \alpha < j < \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha.$$

Therefore, as $j \in \mathbb{N}$, it is the case that

$$\left\lfloor \left\lfloor \frac{j}{\alpha} \right\rfloor \alpha \right\rfloor < j \leq \left\lfloor \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha \right\rfloor. \tag{2.3}$$

Suppose that $j \in \mathcal{B}(\alpha)$. Since

$$\left\lfloor \left\lfloor \frac{j}{\alpha} \right\rfloor \alpha \right\rfloor \quad \text{and} \quad \left\lfloor \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha \right\rfloor$$

are consecutive terms in $\mathcal{B}(\alpha)$, it follows from (2.3) that

$$j = \left\lfloor \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha \right\rfloor.$$

This in turn implies, from (2.2), that

$$0 < \left(1 - \left\{\frac{j}{\alpha}\right\}\right) \alpha < 1.$$

On the other hand, let us suppose that

$$0 < \left(1 - \left\{\frac{j}{\alpha}\right\}\right) \alpha < 1.$$

Then, from (2.2), it follows that

$$\left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha - 1 < j < \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha.$$

Since

$$\left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha - 1 \quad \text{and} \quad \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1\right) \alpha$$

are a pair of irrational numbers whose difference is 1, we have

$$j = \left\lfloor \left(\left\lfloor \frac{j}{\alpha} \right\rfloor + 1 \right) \alpha \right\rfloor .$$

This completes the proof of the lemma, on noting both that this is an element of $\mathcal{B}(\alpha)$ and that the inequality

$$0 < \left(1 - \left\{ \frac{j}{\alpha} \right\}\right) \alpha < 1$$

may be rearranged to give

$$0 < 1 - \frac{1}{\alpha} < \left\{ \frac{j}{\alpha} \right\} .$$

□

Lemma 2.2. *We have* $n \in \mathcal{B}(\phi)$ *if and only if,*

$$\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 2,$$

and $n \in \mathcal{B}(\phi^2)$ *if and only if,*

$$\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 1.$$

Proof. If $n \in \mathcal{B}(\phi)$ then, from Lemma 2.1, we have

$$\left\{ \frac{n}{\phi} \right\} > 1 - \frac{1}{\phi} = \frac{1}{\phi^2}.$$

Note then that

$$\begin{aligned} \left\{ \frac{n}{\phi} \right\} > \frac{1}{\phi^2} &\Longleftrightarrow \{n(\phi - 1)\} > \frac{1}{\phi^2} \\ &\Longleftrightarrow \{n\phi\} > \frac{1}{\phi^2}. \end{aligned}$$

Therefore, if $n \in \mathcal{B}(\phi)$, then

$$1 + \phi > \{n\phi\} + \phi > \frac{1}{\phi^2} + \phi,$$

which implies

$$1 + \phi > \{n\phi\} + \phi > 2,$$

Hence,

$$\begin{aligned} \lfloor (n+1)\phi \rfloor &= \lfloor \lfloor n\phi \rfloor + \{n\phi\} + \phi \rfloor \\ &= \lfloor n\phi \rfloor + \lfloor \{n\phi\} + \phi \rfloor \\ &= \lfloor n\phi \rfloor + 2. \end{aligned}$$

Similarly, if $n \in \mathcal{B}\left(\phi^2\right)$ then, from Lemma 2.1, we obtain

$$\left\{ \frac{n}{\phi^2} \right\} > 1 - \frac{1}{\phi^2} = \frac{1}{\phi},$$

and then

$$\begin{aligned}\left\{\frac{n}{\phi^2}\right\} > \frac{1}{\phi} &\iff \{n(2-\phi)\} > \frac{1}{\phi}\\ &\iff \{-n\phi\} > \frac{1}{\phi}\\ &\iff 1-\{n\phi\} > \frac{1}{\phi}\\ &\iff \{n\phi\} < 1-\frac{1}{\phi} = 2-\phi,\end{aligned}$$

from which we see that

$$\phi < \{n\phi\} + \phi < 2.$$

Therefore,

$$\begin{aligned}\lfloor (n+1)\phi\rfloor &= \lfloor \lfloor n\phi\rfloor + \{n\phi\} + \phi\rfloor\\ &= \lfloor n\phi\rfloor + \lfloor \{n\phi\} + \phi\rfloor\\ &= \lfloor n\phi\rfloor + 1.\end{aligned}$$

The statement of the lemma then follows because $\mathcal{B}(\phi) \cup \mathcal{B}(\phi^2) = \mathbb{N}$. □

Lemma 2.3. *For any $k \in \mathbb{N}$:*

$$2k - \lfloor k\phi\rfloor = \left\lfloor \frac{k}{\phi^2}\right\rfloor + 1.$$

Proof.

$$\begin{aligned}2k - \lfloor k\phi\rfloor &= k - \lfloor k(\phi-1)\rfloor\\ &= k - \left\lfloor \frac{k}{\phi}\right\rfloor\\ &= k + \left\lfloor -\frac{k}{\phi}\right\rfloor + 1\\ &= \left\lfloor k\left(1-\frac{1}{\phi}\right)\right\rfloor + 1\\ &= \left\lfloor \frac{k}{\phi^2}\right\rfloor + 1.\end{aligned}$$

□

3. The Zeckendorf Representation

Theorem 3.1. *The Zeckendorf representation of S_n is given by*

$$F_{\lfloor n\phi\rfloor+1} + \sum_{k=1}^{2n-\lfloor n\phi\rfloor-1} F_{2\lfloor k\phi\rfloor+k-1}.$$

Proof. We start by showing that S_n is equal to the above expression. We then show that this expression is in fact a Zeckendorf representation.

Note first that $2n - \lfloor n\phi\rfloor - 1 = 0$ when $n = 1$ and $n = 2$. In each of these cases the sum on the right is defined to be equal to 0. We now proceed by induction on n. It is easily checked that the statement of the theorem is true for $n = 1, 2$ and 3. Now assume that it is true for some $n \geq 3$. By way of the inductive hypothesis and the definition of S_{n+1}, we have

$$S_{n+1} = F_{\lfloor (n+1)\phi \rfloor} + F_{\lfloor n\phi \rfloor + 1} + \sum_{k=1}^{2n-\lfloor n\phi \rfloor - 1} F_{2\lfloor k\phi \rfloor + k - 1}. \tag{3.1}$$

We deal separately with the cases $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 1$ and $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 2$, beginning with the latter. Indeed, when $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 2$, we have

$$\begin{aligned} F_{\lfloor (n+1)\phi \rfloor} + F_{\lfloor n\phi \rfloor + 1} &= F_{\lfloor n\phi \rfloor + 2} + F_{\lfloor n\phi \rfloor + 1} \\ &= F_{\lfloor n\phi \rfloor + 3} \\ &= F_{\lfloor (n+1)\phi \rfloor + 1} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} 2(n+1) - \lfloor (n+1)\phi \rfloor - 1 &= 2n + 1 - (\lfloor n\phi \rfloor + 2) \\ &= 2n - \lfloor n\phi \rfloor - 1. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) in conjunction with (3.1) then gives

$$S_{n+1} = F_{\lfloor (n+1)\phi \rfloor + 1} + \sum_{k=1}^{2(n+1)-\lfloor (n+1)\phi \rfloor - 1} F_{2\lfloor k\phi \rfloor + k - 1},$$

as required.

Next, consider n such that $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 1$. In this case we obtain

$$\begin{aligned} F_{\lfloor (n+1)\phi \rfloor} + F_{\lfloor n\phi \rfloor + 1} &= 2F_{\lfloor n\phi \rfloor + 1} \\ &= F_{\lfloor n\phi \rfloor + 2} + F_{\lfloor n\phi \rfloor - 1} \\ &= F_{\lfloor (n+1)\phi \rfloor + 1} + F_{\lfloor n\phi \rfloor - 1} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} 2(n+1) - \lfloor (n+1)\phi \rfloor - 2 &= 2n - (\lfloor n\phi \rfloor + 1) \\ &= 2n - \lfloor n\phi \rfloor - 1. \end{aligned} \tag{3.5}$$

Then, using (3.1), (3.4), and (3.5), we have

$$S_{n+1} = F_{\lfloor (n+1)\phi \rfloor + 1} + F_{\lfloor n\phi \rfloor - 1} + \sum_{k=1}^{2(n+1)-\lfloor (n+1)\phi \rfloor - 2} F_{2\lfloor k\phi \rfloor + k - 1}. \tag{3.6}$$

On considering the subscript of the 'extra' term on the right-hand side of (3.6), and that of the 'missing' term in the sum corresponding to $k = 2(n+1) - \lfloor (n+1)\phi \rfloor - 1$, it may be seen that the inductive step will be complete if we show that

$$\lfloor n\phi \rfloor - 1 = 2\lfloor (2(n+1) - \lfloor (n+1)\phi \rfloor - 1)\phi \rfloor + (2(n+1) - \lfloor (n+1)\phi \rfloor - 1) - 1 \tag{3.7}$$

when $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 1$. In this case (3.7) simplifies readily to

$$\lfloor n\phi \rfloor = \lfloor (2n - \lfloor n\phi \rfloor)\,\phi \rfloor + n. \tag{3.8}$$

Using Lemma 2.2, therefore, it suffices to show that (3.8) is true whenever $n = \lfloor m\phi^2 \rfloor$ for some $m \in \mathbb{N}$.

To this end, noting that

$$\lfloor m\phi^2 \rfloor = \lfloor m(1+\phi) \rfloor = m + \lfloor m\phi \rfloor,$$

and using Lemma 2.3 with $k = m + \lfloor m\phi \rfloor$ in the second line below, we find an equation, (3.9), equivalent to (3.8) with $n = m + \lfloor m\phi \rfloor$, as follows:

$$\begin{aligned} \lfloor (m + \lfloor m\phi \rfloor)\phi \rfloor &= \lfloor (2(m + \lfloor m\phi \rfloor) - \lfloor (m + \lfloor m\phi \rfloor)\phi \rfloor)\, \phi \rfloor + m + \lfloor m\phi \rfloor \\ \iff \lfloor (m + \lfloor m\phi \rfloor)\phi - (m + \lfloor m\phi \rfloor) \rfloor &= \left\lfloor \left(\left\lfloor \frac{m + \lfloor m\phi \rfloor}{\phi^2} \right\rfloor + 1 \right) \phi \right\rfloor \\ \iff \lfloor (\phi - 1)(m + \lfloor m\phi \rfloor) \rfloor &= \left\lfloor \left\lfloor \frac{m + \lfloor m\phi \rfloor + \phi^2}{\phi^2} \right\rfloor \phi \right\rfloor \\ \iff \left\lfloor \frac{m + \lfloor m\phi \rfloor}{\phi} \right\rfloor &= \left\lfloor \left\lfloor \frac{m + \lfloor m\phi \rfloor + \phi^2}{\phi^2} \right\rfloor \phi \right\rfloor . \end{aligned} \tag{3.9}$$

Next,

$$\begin{aligned} \frac{m + \lfloor m\phi \rfloor}{\phi} - \lfloor m\phi \rfloor &= \frac{m}{\phi} - \lfloor m\phi \rfloor \left(1 - \frac{1}{\phi} \right) \\ &= \frac{m}{\phi} - \frac{\lfloor m\phi \rfloor}{\phi^2} \\ &= \frac{\{m\phi\}}{\phi^2} \\ &> 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \frac{m + \lfloor m\phi \rfloor}{\phi} - \lfloor m\phi \rfloor - \frac{1}{\phi^2} &= \frac{\{m\phi\} - 1}{\phi^2} \\ &< 0. \end{aligned} \tag{3.11}$$

Results 3.10 and 3.11 show that

$$\left\lfloor \frac{m + \lfloor m\phi \rfloor}{\phi} \right\rfloor = \lfloor m\phi \rfloor. \tag{3.12}$$

Also,

$$\begin{aligned} \left\lfloor \frac{m + \lfloor m\phi \rfloor + \phi^2}{\phi^2} \right\rfloor &= \left\lfloor \frac{m + m\phi - \{m\phi\} + \phi^2}{\phi^2} \right\rfloor \\ &= \left\lfloor \frac{m(1 + \phi) + \phi^2 - \{m\phi\}}{\phi^2} \right\rfloor \\ &= \left\lfloor \frac{m\phi^2 + \phi^2 - \{m\phi\}}{\phi^2} \right\rfloor \\ &= m + \left\lfloor \frac{\phi^2 - \{m\phi\}}{\phi^2} \right\rfloor \\ &= m. \end{aligned}$$

Consequently,

$$\left\lfloor \left\lfloor \frac{m + \lfloor m\phi \rfloor + \phi^2}{\phi^2} \right\rfloor \phi \right\rfloor = \lfloor m\phi \rfloor. \tag{3.13}$$

Results (3.12) and (3.13) show that (3.9) is true, and hence that (3.8) is true whenever $n = \lfloor m\phi^2 \rfloor$ for some $m \in \mathbb{N}$.

Finally, it remains to check that

$$F_{\lfloor n\phi\rfloor+1}+\sum_{k=1}^{2n-\lfloor n\phi\rfloor-1}F_{2\lfloor k\phi\rfloor+k-1}$$

is in fact a Zeckendorf representation. Let us consider first the differences between the subscripts of successive terms in the sum. We have

$$2\lfloor(k+1)\phi\rfloor+(k+1)-1-(2\lfloor k\phi\rfloor+k-1)=2\left(\lfloor(k+1)\phi\rfloor-\lfloor k\phi\rfloor\right)+1,$$

which is equal either to 3 or 5. It is now simply a matter of showing that $\lfloor n\phi\rfloor+1$ is at least 2 larger than the largest subscript arising from the terms in the sum. We have, on using Lemma 2.3 in the third line below,

$$\begin{aligned}\lfloor n\phi\rfloor+1-&(2\lfloor(2n-\lfloor n\phi\rfloor-1)\phi\rfloor+(2n-\lfloor n\phi\rfloor-1)-1)\\&=2\left(\lfloor n\phi\rfloor-n-\lfloor(2n-\lfloor n\phi\rfloor-1)\phi\rfloor\right)+3\\&=2\left(\left\lfloor\frac{n}{\phi}\right\rfloor-\left\lfloor\left\lfloor\frac{n}{\phi^2}\right\rfloor\phi\right\rfloor\right)+3.\end{aligned}$$

This completes the proof of the theorem, on noting that

$$\left\lfloor\frac{n}{\phi}\right\rfloor=\left\lfloor\frac{n\phi}{\phi^2}\right\rfloor\geq\left\lfloor\left\lfloor\frac{n}{\phi^2}\right\rfloor\phi\right\rfloor.$$

□

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