CLOSED FORM EVALUATIONS OF SOME SERIES COMPRISING SUMS OF EXPONENTIATED MULTIPLES OF TWO-TERM AND THREE-TERM CATALAN NUMBER LINEAR COMBINATIONS

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ABSTRACT. Closed form evaluations of some infinite series comprising sums of exponentiated multiples of two-term and three-term Catalan number linear combinations are presented using three contrasting approaches. Known power series expansions of the trigonometric functions $\sin(4\alpha)$ and $\sin(6\alpha)$ each readily give a set of (four) results which are re-formulated via a hypergeometric route and, additionally, using only the generating function for the Catalan sequence; the latter two methods are shown to be connected.

1. INTRODUCTION

Let $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ be the Catalan sequence, with $c_n = \frac{1}{n+1}\binom{2n}{n}$ the closed form of its $(n+1)$th term ($n \geq 0$) and (ordinary) generating function

$$G(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_{n \geq 0} c_n x^n \tag{1.1}$$

familiar as a solution to the quadratic equation

$$xG^2(x) - G(x) + 1 = 0. \tag{1.2}$$

This paper continues the theme of [2] and presents new closed form evaluations of infinite series comprising certain sums of exponentiated multiples of two-term and three-term Catalan number linear combinations, in that order. The first set of (four) results is obtained conveniently, and very simply, as a direct consequence of a known expansion (in odd powers of $\sin(\alpha)$) of the trigonometric function $\sin(4\alpha)$, after which two alternative formulations are given—first from the hypergeometric consideration of a generalized sum whose particular instances we seek, and then more directly by appeal to the Catalan sequence generating function. The derivation process is repeated for a further set of identities in which a three-term Catalan number combination appears instead, the series evaluations necessarily described only in brief. Those series examined here and previously are but cases lying within a complete class of series whose general form is known but for which theory leading to its evaluation has yet to be produced; this, at present, poses an open problem.

Four series evaluations—of similar type but of simpler structure—were detailed in [2] through an expansion of $\sin(2\alpha)$ and also, separately, using the known form of $G(x)$ in an elementary manner. The reader will see that the increased level of complexity of the results here in this paper is reflected more so in the hypergeometric routes employed. Collectively all three contrasting evaluation techniques offer analysis which is interesting, and they provide, of course, a means of cross-verification. Two of them can in fact be connected, and this is explained after the results sections.

On a more general point, we mention that the wider class of series expansions for $\sin(2p\alpha)$ (integer $p \geq 1$) has its roots in the 18th century geometric work of a Chinese scholar, Antu Ming, through which the first discovery of the Catalan numbers was made. This author has written extensively on the subject, and the interested reader is referred to [1], and to other citations in another related publication [3], for further information.

2. Results Set 1

2.1. Method I: Evaluation of Known Series. The aforementioned series expansion of $\sin(4\alpha)$ is (see, for example, [1, Result II, p. 42] or [3, Eq. (1.3), p. 236])

$$\sin(4\alpha)/2 = 2\sin(\alpha) - 5\sin^3(\alpha) + \sum_{n=1}^{\infty}\left[\frac{8c_{n-1} - c_n}{4^n}\right]\sin^{2n+3}(\alpha); \tag{2.1}$$

it is the appearance of the linear combination $8c_{n-1} - c_n$ of Catalan terms that forms an essential element of our first set of series evaluations. Substituting values $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ and $\frac{\pi}{6}$ into (2.1) delivers, after a little work in each case, respective results as follows:

$$\begin{aligned}
\sum_{n\ge1}(1/4)^n[8c_{n-1} - c_n] &= 3,\\
\sum_{n\ge1}(3/16)^n[8c_{n-1} - c_n] &= \frac{5}{3},\\
\sum_{n\ge1}(1/8)^n[8c_{n-1} - c_n] &= 1,\\
\sum_{n\ge1}(1/16)^n[8c_{n-1} - c_n] &= 2\sqrt{3} - 3.
\end{aligned} \tag{2.2}$$

These values of α each lie within a range of convergence $|\alpha| \le \frac{\pi}{2}$ which holds for all series expansions of $\sin(2p\alpha)$ (of which (2.1) gives that for $p = 2$), the issue of convergence in the general case motivating the article [3] where the natural principal range of convergence $|\alpha| < \frac{\pi}{2}$ was extended to $|\alpha| \le \frac{\pi}{2}$.

The above, then, is the set of four results that emerge naturally from the expansion (2.1) and which, as described, we now reformulate by means of two other methods.

2.2. Method II: Hypergeometric Formulation. We begin by considering a generalized sum $T(\beta)$ of form

$$T(\beta) = \sum_{n=1}^{\infty}\beta^n[8c_{n-1} - c_n] \tag{2.3}$$

for $\beta \ne 0$, with (2.2) comprising those evaluations for values $\beta = \frac{1}{4}, \frac{3}{16}, \frac{1}{8}$ and $\frac{1}{16}$. The relation $c_{n+1} = [2(2n+1)/(n+2)]c_n$ between neighboring terms of the Catalan sequence allows us to write[1]

$$T(\beta) = 2\sum_{n=1}^{\infty}\beta^n\left(\frac{2n+5}{n+1}\right)c_{n-1}, \tag{2.4}$$

and the form of the summand now lends itself to a straightforward conversion

$$T(\beta) = 7\beta\,{}_3F_2\left(\left.\begin{matrix}\frac{9}{2}, \frac{1}{2}, 1\\ 3, \frac{7}{2}\end{matrix}\,\right|\,4\beta\right) \tag{2.5}$$

[1]Thus, the series of (2.2) could each have been expressed as sums of exponentiated multiples of single Catalan numbers, as in [2], but inclusive now of an additional rational (coefficient) term involving the summing index.

of $T(\beta)$ to a hypergeometric representation by hand—this is the basis of our formulation here, in combination with a result whose proof is given in Appendix A (the author is indebted to Professor Ira Gessel for his proof outline which was communicated privately).

Lemma 2.1. *The following hypergeometric identity holds for all values of x:*

$$ {}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2}\right) = \frac{1}{7}(1+x)(7-x). $$

In the first instance we seek

$$ T\left(\frac{1}{4}\right) = \frac{7}{4}\,{}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \,\middle|\, 1\right). \tag{2.6} $$

To evaluate the ${}_3F_2(1)$ series then, in view of Lemma 2.1, we set $1 = 4x/(1+x)^2$ which returns just one (repeated) solution $x = 1$ and gives $T(\frac{1}{4}) = \frac{7}{4} \cdot \frac{1}{7}(1+1)(7-1) = 3$ as required.

Secondly, we look for

$$ T\left(\frac{3}{16}\right) = \frac{21}{16}\,{}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \,\middle|\, \frac{3}{4}\right). \tag{2.7} $$

Putting $3/4 = 4x/(1+x)^2$ and using the solution $x = \frac{1}{3}$ (rather than $x = 3$) we see that Lemma 2.1 gives $T(\frac{3}{16}) = \frac{21}{16} \cdot \frac{1}{7}(1+\frac{1}{3})(7-\frac{1}{3}) = \frac{5}{3}$.

We leave the third result of (2.2) as a small reader exercise, and finish with the final one which necessitates the evaluation of

$$ T\left(\frac{1}{16}\right) = \frac{7}{16}\,{}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \,\middle|\, \frac{1}{4}\right). \tag{2.8} $$

The roots of the equation $1/4 = 4x/(1+x)^2$ are $x = 7 \pm 4\sqrt{3}$, and it is $x = 7 - 4\sqrt{3}$ which correctly delivers $T(\frac{1}{16}) = \frac{7}{16} \cdot \frac{1}{7}[1 + (7-4\sqrt{3})][7 - (7-4\sqrt{3})] = 2\sqrt{3} - 3$.

2.3. **Method III: Generating Function Formulation.**

We write, for $\beta \neq 0$,

$$ \begin{aligned} T(\beta) &= \sum_{n=1}^{\infty} \beta^n [8c_{n-1} - c_n] \\ &= 8\sum_{n=1}^{\infty} \beta^n c_{n-1} - \sum_{n=1}^{\infty} \beta^n c_n \\ &= 8\beta\sum_{n=0}^{\infty} \beta^n c_n - \sum_{n=1}^{\infty} \beta^n c_n \\ &= 8\beta\sum_{n=0}^{\infty} \beta^n c_n - \left(\sum_{n=0}^{\infty} \beta^n c_n - 1\right) \\ &= (8\beta - 1)G(\beta) + 1 \end{aligned} \tag{2.9} $$

in terms of the generating function for the Catalan sequence. Noting from (1.1) that $G(\frac{1}{4}) = 2$, $G(\frac{3}{16}) = \frac{4}{3}$ and $G(\frac{1}{16}) = 8(1 - \sqrt{3}/2)$, our required results follow immediately:

$$\begin{aligned} T(1/4) &= G(1/4) + 1 = 2 + 1 = 3, \\ T(3/16) &= (1/2)G(3/16) + 1 = (1/2) \cdot (4/3) + 1 = 5/3, \\ T(1/8) &= 1, \\ T(1/16) &= 1 - (1/2)G(1/16) = 1 - (1/2) \cdot 8(1 - \sqrt{3}/2) = 2\sqrt{3} - 3. \end{aligned} \tag{2.10}$$

We finish the section with a word on convergence. As β assumes values $\beta = \frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}$ (corresponding to $\alpha = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ in the series (2.1), and moving from a more central position within the convergence interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to the edge), the convergence rates of the series (2.2) decrease as the listing is read upwards—exponentially so between the final two; this latter behavior is to be expected (see [2, Remark 2, p. 119] for an explanation in the context of results there), and is exhibited also in the suite of evaluations given below.

3. Results Set 2

3.1. **Method I: Evaluation of Known Series.**

Our starting point is the series expansion [1, p. 43]

$$\begin{aligned} \sin(6\alpha)/2 = {} & 3\sin(\alpha) - (35/2)\sin^3(\alpha) + (189/8)\sin^5(\alpha) \\ & - \sum_{n=1}^{\infty} \left[\frac{256c_{n-1} - 64c_n + 3c_{n+1}}{2^{2n+3}} \right] \sin^{2n+5}(\alpha) \end{aligned} \tag{3.1}$$

for $\sin(6\alpha)$, into which we again substitute $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ and $\frac{\pi}{6}$; this leads eventually to the following (resp.) series evaluations:

$$\begin{aligned} \sum_{n\geq 1} (1/4)^n [256c_{n-1} - 64c_n + 3c_{n+1}] &= 73, \\ \sum_{n\geq 1} (3/16)^n [256c_{n-1} - 64c_n + 3c_{n+1}] &= 45, \\ \sum_{n\geq 1} (1/8)^n [256c_{n-1} - 64c_n + 3c_{n+1}] &= 5 + 16\sqrt{2}, \\ \sum_{n\geq 1} (1/16)^n [256c_{n-1} - 64c_n + 3c_{n+1}] &= 13. \end{aligned} \tag{3.2}$$

3.2. **Method II: Hypergeometric Formulation.**

We write

$$\begin{aligned} U(\beta) &= \sum_{n=1}^{\infty} \beta^n [256c_{n-1} - 64c_n + 3c_{n+1}] \\ &= 198\beta\, {}_3F_2 \left(\begin{matrix} \frac{13}{2}, \frac{1}{2}, 1 \\ 4, \frac{9}{2} \end{matrix} \middle| \, 4\beta \right) \end{aligned} \tag{3.3}$$

in hypergeometric form ($\beta \neq 0$). The evaluations of $U(\frac{1}{4}), U(\frac{3}{16}), U(\frac{1}{8})$ and $U(\frac{1}{16})$ are made in an analogous fashion to that seen in Section 2.2, this time utilizing a different requisite lemma (whose proof, likewise, follows the same line of argument and is left as an exercise: after some algebra (and with reference to Appendix A for notation) it is seen to reduce simply to showing that, for $n \geq 0$, $[y^n]\{256G(y) - 61G^2(y) + 3G^3(y)\} = 198 \cdot 4^n (\frac{13}{2})_n (\frac{1}{2})_n / [(4)_n (\frac{9}{2})_n]$ which is straightforward, though tedious, by hand).

Lemma 3.1. *The following hypergeometric identity holds for all values of x:*

$$ {}_3F_2\left(\begin{matrix} \frac{13}{2}, \frac{1}{2}, 1 \\ 4, \frac{9}{2} \end{matrix} \,\middle|\, \frac{4x}{(1+x)^2} \right) = \frac{1}{198}(1+x)(198 - 55x + 3x^2). $$

3.3. **Method III: Generating Function Formulation.** Consider, for $\beta \neq 0$, a general sum

$$ F(\beta; a, b, c) = \sum_{n=1}^{\infty} \beta^n [a c_{n-1} + b c_n + c c_{n+1}] = a \sum_{n=1}^{\infty} \beta^n c_{n-1} + b \sum_{n=1}^{\infty} \beta^n c_n + c \sum_{n=1}^{\infty} \beta^n c_{n+1}, \quad (3.4) $$

with a, b, c arbitrary constants. Rewriting the individual sums as

$$ \begin{aligned} \sum_{n=1}^{\infty} \beta^n c_{n-1} &= \beta \sum_{n=0}^{\infty} \beta^n c_n = \beta G(\beta), \\ \sum_{n=1}^{\infty} \beta^n c_n &= \sum_{n=0}^{\infty} \beta^n c_n - 1 = G(\beta) - 1, \\ \sum_{n=1}^{\infty} \beta^n c_{n+1} &= \beta^{-1} \sum_{n=2}^{\infty} \beta^n c_n \\ &= \beta^{-1} \left(\sum_{n=0}^{\infty} \beta^n c_n - 1 - \beta \right) = \beta^{-1}[G(\beta) - (1+\beta)], \end{aligned} \quad (3.5) $$

then (3.4) reads

$$ F(\beta; a, b, c) = (a\beta + b + c\beta^{-1})G(\beta) - (b + c) - c\beta^{-1}, \quad (3.6) $$

so that

$$ U(\beta) = F(\beta; 256, -64, 3) = (256\beta - 64 + 3\beta^{-1})G(\beta) + 61 - 3\beta^{-1}, \quad (3.7) $$

from which correct values for $U(\frac{1}{4}), U(\frac{3}{16}), U(\frac{1}{8})$ and $U(\frac{1}{16})$ are duly delivered as the reader is invited to check (note that the function $256\beta - 64 + 3\beta^{-1}$ is identically zero at $\beta = \frac{3}{16}$ and $\frac{1}{16}$ so that $G(\frac{3}{16})$ and $G(\frac{1}{16})$ are not required; $G(\frac{1}{8}) = 4(1 - 1/\sqrt{2})$ is needed here, however, as seen in the example below).

Example. We illustrate further Methods II and III with an example from one of the identities (3.2)—that for $U(\frac{1}{8})$. Using the value $x = 3 - 2\sqrt{2}$ (a solution of $4x/(1+x)^2 = \frac{1}{2}$), then by (3.3) and Lemma 3.1 Method II offers the evaluation $U(\frac{1}{8}) = \frac{198}{8}\,{}_3F_2(\frac{13}{2}, \frac{1}{2}, 1; 4, \frac{9}{2}|\frac{1}{2}) = \frac{198}{8} \cdot \frac{1}{198}[1 + (3 - 2\sqrt{2})][198 - 55(3 - 2\sqrt{2}) + 3(3 - 2\sqrt{2})^2] = 5 + 16\sqrt{2}$ when simplified. This value is given by (3.7) according to Method III, on the other hand, as $U(\frac{1}{8}) = [256(\frac{1}{8}) - 64 + 3(8)]G(\frac{1}{8}) + 61 - 3(8) = 37 - 8G(\frac{1}{8}) = 37 - 8 \cdot 4(1 - 1/\sqrt{2}) = 5 + 16\sqrt{2}$.

Remark 3.1. $T(\beta)$ is the instance $T(\beta) = F(\beta; 8, -1, 0)$, in which case (3.6) reads as (2.9). In [2] the sum $S(\beta) = \sum_{n=1}^{\infty} \beta^n c_{n-1}$ was considered, with $S(\beta) = \beta G(\beta)$ recovered trivially as $S(\beta) = F(\beta; 1, 0, 0)$ by (3.6).

Remark 3.2. We remark that the forerunner article [2] gives only two evaluation methods for the sum $S(\beta) = \sum_{n=1}^{\infty} \beta^n c_{n-1}$, corresponding to Methods I and III here. Due to the very simple nature of $S(\beta)$ a hypergeometric approach was felt unnecessary (note, however, that the result $S(\frac{1}{4})(= \frac{1}{4}G(\frac{1}{4})) = \frac{1}{2}$ can be reproduced directly as a stand-alone one by applying a routine hypergeometric result, as alluded to in Remark 1 therein—see Appendix B here). For consistency, we state the counterpart to Lemmas 2.1 and 3.1 which could have been applied—this may be written as ${}_2F_1(\frac{1}{2},1;2|4x/(1+x)^2) = G(x/(1+x)^2) = 1+x$, and is immediate from Appendix A (note the term $1+x$ common to all three results).

Remark 3.3. We know that convergence of the infinite series $T(\beta)$ and $U(\beta)$ (and $S(\beta)$ examined in [2]) is guaranteed over the interval $\beta \in (0, \frac{1}{4}]$. Aside from the four values $\beta = \frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}$ of particular interest, other evaluations can of course be made, examples of which are $T(\frac{1}{32}) = 3\sqrt{14} - 11$, $T(\frac{5}{32}) = \frac{1}{5}(9-\sqrt{6})$, $U(\frac{3}{32}) = \frac{1}{3}(32\sqrt{10}-41)$ and $U(\frac{7}{32}) = \frac{1}{49}(2957 - 160\sqrt{2})$ (together with, for instance, $S(\frac{9}{128}) = \frac{1}{16}(8-\sqrt{46})$ and $S(\frac{21}{128}) = \frac{1}{16}(8-\sqrt{22})$). These, and those of (2.2) and (3.2), have all been confirmed numerically by calculating actual sums to a high degree of computational convergence accuracy.

Remark 3.4. Graphs of $S(\beta), T(\beta)$ and $U(\beta)$ are (increasing) monotonic ones, and over a relatively small β range exhibit strong linearity. Accordingly, data points generated for $\beta \in [\frac{1}{128}, \frac{3}{32}]$ result in the following relations via linear regression: $S(\beta) = 1.1249\beta - 0.0024$, $T(\beta) = 7.7357\beta - 0.0142$, $U(\beta) = 215.1569\beta - 0.3297$. Linearity strength is confirmed by a minimum value of 0.9995 for the Pearson correlation coefficient across all three data fits.

4. Connecting Methods II and III

We end the paper with an observation which enables Methods II and III to be linked, and demonstrate this in relation to the sum $T(\beta)$. The perceptive reader might have noticed that the values of x used in the applications of Lemma 2.1 to evaluate $T(\beta)$ are, in each case, linked to those corresponding values of β (from which they have been calculated) according to simply $x(\beta) = G(\beta) - 1$. For example, $x(\frac{1}{4}) = G(\frac{1}{4}) - 1 = 2 - 1 = 1$, $x(\frac{3}{16}) = G(\frac{3}{16}) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$, and so on. This insight follows from the proof methodology adopted for Lemma 2.1 and means that, while Methods II and III *are* different, the evaluating formulas (2.5) and (2.9) are connected through Lemma 2.1 and we can in principle move from one to the other; this proves to be so using merely the quadratic equation (1.2) governing $G(x)$, as illustrated: we write, beginning with (2.5), $T(\beta) = 7\beta\, {}_3F_2(\frac{9}{2}, \frac{1}{2}, 1; 3, \frac{7}{2}|4\beta) = 7\beta\, {}_3F_2(\frac{9}{2}, \frac{1}{2}, 1; 3, \frac{7}{2}|4x(\beta)/(1+x(\beta))^2) = 7\beta \cdot \frac{1}{7}[1+x(\beta)][7-x(\beta)] = \beta G(\beta)[8-G(\beta)] = 8\beta G(\beta) - \beta G^2(\beta) = 8\beta G(\beta) - [G(\beta) - 1]$ (by (1.2)) $= (8\beta - 1)G(\beta) + 1$, which is (2.9).

In a similar fashion the expression (3.7) for $U(\beta)$, though requiring more algebra, can be deduced directly from (3.3) via Lemma 3.1 (the details of which are omitted here but set out in Appendix C for completeness).

5. Summary

Following on from previous work [2], further closed form evaluations of some series involving Catalan numbers have been detailed. The two sets of results presented are new, and each has been formulated using three different approaches which the author hopes will be of interest along with the series evaluations themselves—on the latter, one feature of the eight closed forms across (2.2) and (3.2) is the mixture of integers, irrationals and a rational.

The evaluation of a p-term sum $\sum_{n\geq 1} \beta^n [a_0^{(p)} c_{n-1} + a_1^{(p)} c_n + a_2^{(p)} c_{n+1} + \cdots + a_{p-1}^{(p)} c_{n+p-2}]$ defining a general class of series (of which this paper and [2] address the cases $p = 1, 2, 3$) is a natural one to examine in the future, as mentioned briefly in the Introduction. As a piece of extension analysis it certainly looks to be non-trivial, with Methods II and III each contenders for a solution to this broader problem.

Dedication

This article is dedicated to the memory of David R. French ('Frenchy') who passed away February 2014, at the age of 71, after a prolonged period of illness. A good friend, a great colleague and an inspirational educator, sadly missed.

Appendix A

Here we prove Lemma 2.1.

Proof. Let $y = x/(1+x)^2$. The resulting quadratic equation $0 = yx^2 + (2y-1)x + y$ in $x(y)$ has solutions $x(y) = (1 \pm \sqrt{1-4y})/2y - 1$. Taking the negative sign for the radical we use $x(y) = (1 - \sqrt{1-4y})/2y - 1 = G(y) - 1$. Thus, Lemma 2.1 holds if

$$ {}_3F_2\left(\begin{matrix} \frac{9}{2}, \frac{1}{2}, 1 \\ 3, \frac{7}{2} \end{matrix} \middle| \, 4y \right) = \frac{1}{7}[1 + (G(y)-1)][7 - (G(y)-1)] = \frac{1}{7}G(y)[8 - G(y)], \tag{P.1} $$

which is to say (equating coefficients of y^n across the equation), for $n \geq 0$,

$$ [y^n]\{8G(y) - G^2(y)\} = 7 \cdot 4^n \frac{(\frac{9}{2})_n (\frac{1}{2})_n}{(3)_n (\frac{7}{2})_n}, \tag{P.2} $$

where the rising factorial function (with $(w)_0 = 1$)

$$ (w)_k = w(w+1)(w+2)(w+3)\cdots(w+k-1) \tag{P.3} $$

has been used in standard fashion (integer $k \geq 0$). Now it is well-known that, for positive integer r, the rth power of the Catalan sequence generating function $G(x)$ takes the power series form

$$ G^r(x) = \sum_{n=0}^{\infty} \frac{r}{2n+r} \binom{2n+r}{n} x^n, \tag{P.4} $$

so that we can write

$$ \begin{aligned} [y^n]\{8G(y) - G^2(y)\} &= 8\frac{1}{2n+1}\binom{2n+1}{n} - \frac{2}{2n+2}\binom{2n+2}{n} \\ &= 2(2n+7)\frac{(2n)!}{n!(n+2)!}, \end{aligned} \tag{P.5} $$

from which (P.2) follows readily, as required, using the sub-results $(\frac{9}{2})_n/(\frac{7}{2})_n = (2n+7)/7$, $(3)_n = (n+2)!/2$ and $(\frac{1}{2})_n = 4^{-n}(2n)!/n!$ as appropriate in (P.5). $\square$

We remark, by way of a point of interest, that the idea behind consideration of the ${}_3F_2(\frac{9}{2}, \frac{1}{2}, 1; 3, \frac{7}{2}|4x/(1+x)^2)$ hypergeometric series of Lemma 2.1 is motivated by the fact that (i) the upper parameter 9/2 and lower parameter 7/2 'almost' cancel, and if they were removed the resulting ${}_2F_1$ series would have parameters close to those of the hypergeometric form of the Catalan sequence generating function $G(x) = {}_2F_1(\frac{1}{2}, 1; 2|4x)$, and (ii) $G(x)$ has the property that $G(x/(1+x)^2) = 1 + x$.

Appendix B

In this appendix we evaluate $S(\frac{1}{4})$ by appeal to a well-known theorem of Gauss. The result states that any ${}_2F_1$ hypergeometric function (with possible complex parameters and argument) of general form

$$ {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z\right) \tag{B.1}$$

converges for $|z| < 1$, and moreover at the radius of convergence $z = 1$ it converges if and only if $\mathrm{Re}\{b_1 - (a_1 + a_2)\} > 0$ to a value

$$\frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)} \tag{B.2}$$

given in terms of the Gamma function. With $S(\beta) = \beta\, {}_2F_1(\frac{1}{2}, 1; 2|4\beta)$ we thus have (noting convergence is assured since $\mathrm{Re}\{2 - (\frac{1}{2} + 1)\} = \mathrm{Re}\{\frac{1}{2}\} = \frac{1}{2} > 0$) $S(\frac{1}{4}) = \frac{1}{4}\, {}_2F_1(\frac{1}{2}, 1; 2|1) = \frac{1}{4} \cdot \Gamma(2)\Gamma(\frac{1}{2})/[\Gamma(\frac{3}{2})\Gamma(1)]$. Writing $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$, and noting that $\Gamma(2) = 1! = 1 = 0! = \Gamma(1)$, then $S(\frac{1}{4}) = \frac{1}{4}/\frac{1}{2} = \frac{1}{2}$.

Appendix C

Here we establish the connection between those evaluating formulas of Sections 3.2 and 3.3 (under Methods II and III) for the general series $U(\beta)$.

Consider, from (3.3), $U(\beta) = 198\beta\, {}_3F_2(\frac{13}{2}, \frac{1}{2}, 1; 4, \frac{9}{2}|4\beta) = 198\beta\, {}_3F_2(\frac{13}{2}, \frac{1}{2}, 1; 4, \frac{9}{2}|\frac{4x(\beta)}{(1+x(\beta))^2}) =$ (by Lemma 3.1) $198\beta \cdot \frac{1}{198}[1 + x(\beta)][198 - 55x(\beta) + 3x^2(\beta)] = \cdots = 256\beta G(\beta) - 61\beta G^2(\beta) + 3\beta G^3(\beta)$. Now, using (1.2), we replace $\beta G^2(\beta)$ with $G(\beta) - 1$, and so in turn write $\beta G^3(\beta) = G(\beta)[\beta G^2(\beta)] = G(\beta)[G(\beta) - 1] = G^2(\beta) - G(\beta) = \frac{G(\beta)-1}{\beta} - G(\beta) = (\frac{1}{\beta} - 1)G(\beta) - \frac{1}{\beta}$, giving $U(\beta) = 256\beta G(\beta) - 61[G(\beta) - 1] + 3[(\frac{1}{\beta} - 1)G(\beta) - \frac{1}{\beta}] = (256\beta - 64 + \frac{3}{\beta})G(\beta) + 61 - \frac{3}{\beta}$, which is (3.7).

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