COEFFICIENT CONVERGENCE OF RECURSIVELY DEFINED POLYNOMIALS

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Abstract. This article partially answers an open problem of Kimberling. Consider a sequence of polynomials satisfying an mth order recursive relation with polynomial coefficients. Under what circumstances can we say anything exact about the coefficients of x^i? The paper's main theorem asserts that under modest assumptions, there exists a computable constant, c, such that, for each i, the coefficients of x^i eventually satisfy a polynomial of degree i with the ith difference operator applied to this polynomial equaling c^i.

1. Introduction

Kimberling [2] studies the polynomial sequence satisfying the polynomial recursion

$$G_n(x) = (ax+b)G_{n-1}(x) + (cx^2+dx+e)G_{n-2}(x) + (fx+g), \tag{1.1}$$

with

$$G_0(x) = 1, G_1(x) = 1+x, \qquad a \neq 0, b = 0, e = 0. \tag{1.2}$$

Kimberling shows two convergence results. Letting

$$G_n(x) = \sum_{i=0}^{n} g_n^{(i)} x^i, \tag{1.3}$$

we have both *pointwise* convergence,

$$\lim_{x\to\infty} G_n(x) = \frac{g+fx}{1-(a+d)x-cx^2},$$

as well as *coefficient* convergence

$$\lim_{n\to\infty} g_n^{(i)} = \text{ coefficient of } x^i \text{ in } \frac{g+fx}{1-(a+d)x-cx^2}.$$

Kimberling poses two questions: (1) Can the assumptions on a, b, e in (1.2) be changed? (2) The order of the recursion in (1.1) is 2; can it be raised?

In this paper we prove the following main theorem.

Theorem 1.1. *Suppose the polynomial sequence* $\{G_n(x)\}_{n\geq 0}$ *satisfies the recursion*

$$G_n(x) = \sum_{i=1}^{m} p_i(x) G_{n-i}(x), \qquad m \geq 2, \tag{1.4}$$

with

$$p_i(x) = \sum_{j=0}^{i} c_i^{(j)} x^j, \qquad 1 \leq i \leq m, \tag{1.5}$$

with initial conditions

$$G_i(x) = \sum_{j=0}^{i} x^j, \qquad 0 \leq i \leq m-1, \tag{1.6}$$

and with the following assumptions

$$c_1^{(0)} = 1, c_i^{(0)} = 0, \qquad 1 < i \leq m. \tag{1.7}$$

Then, using (1.3) *we have*
(i)

$$g_n^{(0)} = 1, \qquad n \geq 1. \tag{1.8}$$

(ii) The array $\{g_n^{(i)}\}_{n,i \geq 0}$ *has triangular support, that is,*

$$g_n^{(i)} \neq 0 \text{ implies } 0 \leq i \leq n. \tag{1.9}$$

Define the diagonal sequence by

$$D_i = g_i^{(i)}. \tag{1.10}$$

Then
(iii)

$$D_n = \sum_{k=1}^{m} c_k^{(k)} D_{n-k}. \tag{1.11}$$

(iv) For each fixed i, *for sufficiently large* n, *we have*

$$\Delta^i g_n^{(i)} = \left(\sum_{j=1}^{m} c_j^{(1)} \right)^i, \tag{1.12}$$

with Δ *the finite difference operator with respect to the variable* n [1].

Equation (1.12) implies the main result of this paper.

Corollary 1.2. *With assumptions* (1.4)–(1.7)*, we have for each fixed* i *that the coefficient sequence* $\{g_n^{(i)}\}_{n,i \geq 0}$ *eventually, that is for all* $n \geq n_0$*, exactly satisfies a degree* i *polynomial in the variable* n*.*

It might be worthwhile to compare the assumptions and conclusions in this paper and in [2]. A comparison of assumptions is presented in Table 1. A comparison of conclusions is presented in Table 2. Notice that some of the results of [2] are lost in this paper. The basic point of this paper is that interesting patterns do exist in the coefficients even if there is no limiting generating function.

An outline of the rest of this paper is the following. In the next section, we illustrate the main theorem with a particular example. We also discuss generating functions and pointwise convergence. In the final section, we prove the main theorem.

Table 1. A comparison of assumptions in [2] and this paper.

Assumption	Kimberling	This paper
$p_0(x)$	$\neq 0$	0
$c_1^{(0)}$	0	1
$c_1^{(1)}$	$\neq 0$	Unrestricted
$c_2^{(0)}$	0	0

Table 2. A comparison of conclusions in [2] and this paper.

Conclusion	Kimberling	This paper
Coefficient convergence	To a constant	To a polynomial value
Pointwise convergence	Yes	No
Limiting coefficient values	Recursive sequence	Difference operator pattern

2. An Example

We illustrate the main theorem and associated concepts with the following example [3].

Example 2.1. *Define a polynomial sequence,* $\{G_n\}_{n\geq 0}$*, by*

$$G_n(x) = p_1(x)G_{n-1}(x) + p_2(x)G_{n-2}(x), \tag{2.1}$$

with

$$p_1(x) = 1 - 2x, p_2(x) = x - x^2, G_0(x) = 1, G_1(x) = 1 + x.$$

It is easy to derive that

$$G_n(x) = (1 + 2x)(1 - x)^n - 2x(-x)^n, \qquad n \geq 0. \tag{2.2}$$

This can be verified by first checking the initial conditions for $n = 1, 2$ and then substituting (2.2) into (2.1) and simplifying.

Table 3 presents the first few rows. By expanding (2.2) we see that

$$g_n^{(i)} = (-1)^i \frac{1}{i!}(n - (3i - 1))(n)_{(i-1)}, \qquad n \geq i. \tag{2.3}$$

So in this particular example the *eventually* in Corollary 1.2 means for $n \geq i$. Note that the word *eventually* is needed in Corollary 1.2, since for $1 \leq n < i$, (2.3) does not uniformly yield 0 as it should.

We can use this example to illustrate the main theorem.

(i) By (2.2), $g_n^{(0)} = 1, \qquad n \geq 0.$

(ii) Again using (2.2), $g_n^{(i)} \neq 0$ implies $0 \leq i \leq n$.

(iii) By (2.3) and (1.10), $D_i = -(-1)^i(2i - 1)$. And indeed, these alternating odds satisfy

$$D_i = -2D_{i-1} - D_{i-2}$$

as required.

(iv) Equation (1.12) is implied by the even stronger assertion that

$$\Delta g_n^{(i+1)} = -g_n^{(i)}, \qquad i \geq 0. \tag{2.4}$$

TABLE 3. $G_n(x), 0 \leq n \leq 4$, with $G_n(x)$ defined by (2.2). For example $G_2(x) = 1 - 3x^2$.

$G_n(x)$	Constant Coef.	Coef. of x	Coef. of x^2	Coef. of x^3	Coef. of x^4
$G_0(x)$	1				
$G_1(x)$	1	1			
$G_2(x)$	1	0	-3		
$G_3(x)$	1	-1	-3	5	
$G_4(x)$	1	-2	-2	8	-7

It is easy to see that (2.4) implies (1.12), since by applying Δ to both sides of (2.4) i times one arrives at (1.12) by (1.8).

Expanding (2.4) we obtain

$$g_{n+1}^{(i+1)} - g_n^{(i+1)} = -g_n^{(i)}. \tag{2.5}$$

We now proceed to prove (2.5).

Proof. We first substitute (2.3) into (2.5). We can then make the following simplifications.

- The minus signs on both sides cancel.
- After canceling factorials in the denominators we are left with a factor of $i+1$ on the right side.
- We can cancel a common factor of $(n)_{i-1}$ from both sides.

We then see that to prove (2.5) we must equivalently prove

$$(n+1)(n+1-(3i+2)) - (n-(i-1))(n-(3i+2)) = (i+1)(n-(3i-1)). \tag{2.6}$$

But for each fixed n, (2.6) is a degree 2 polynomial in i and hence for each fixed n to prove (2.6), it suffices to show (2.6) is true at 3 points. Since for each fixed n, the resulting equation is a degree 2 polynomial in i, it is simply a matter of verification. However, computations are easiest at $n = 3i+1$, $n = 3i+2$, and $n = 3i-1$, since one of the three terms vanish. For example, if we substitute $n = 3i+1$ then (2.6) reduces to verification that $2i+2 = 2(i+1)$. This completes the proof. □

We use this example to lightly discuss why the obvious generating function approach used in [2] is not useful to prove the main theorem. As can easily be seen from (2.3) or (2.2), as n goes to infinity the coefficients blow up. So there is no limit function. The pointwise limit of $G_n(x)$ also shows this. For example, the sequence is diverging by oscillation at $x = 1$. The pointwise limit (where it exists) has several discontinuities. Where the limit exists it is not uniform. For example, while $G_n(0) = 1$ for all n, nevertheless, $\lim_{n\to\infty} G_n(\frac{1}{n}) = e^{-1}$. Because of these considerations, we will employ a difference operator approach in the proof of the main theorem in the next section.

It is natural to ask whether this example can be generalized. Are there other examples where (2.4) holds? It turns out that (2.1) is a member of a one-parameter family of such examples.

Lemma 2.2. *Suppose* (1.4)–(1.7) *hold with* $m = 2$*. Then necessary and sufficient conditions for* (2.4) *to hold are the following:*

$$c_1^{(1)} = b, \qquad c_2^{(2)} = 1 + b, \qquad c_2^{(1)} = -1 - b. \tag{2.7}$$

Proof. Both necessity and sufficiency require the following identity which follows from (1.3), (1.5), (1.4), and (1.7):

$$g_n^{(i)} = g_{n-1}^{(i)} + c_1^{(1)} g_{n-1}^{(i-1)} + c_2^{(1)} g_{n-2}^{(i-1)} + c_2^{(2)} g_{n-2}^{(i-2)}. \tag{2.8}$$

We now proceed to prove necessity and sufficiency.

Necessity. We assume (2.4) holds with $m = 2$ and proceed to derive (2.7).

Proof that $c_2^{(1)} = -1 - b$**.** Repeatedly applying the Δ operator to (2.4), we obtain $\Delta^2 g_n^{(i)} = (-1)^2 g_n^{(i-2)}$, and similarly $\Delta^3 g_n^{(i)} = (-1)^3 g_n^{(i-3)}$, so that by induction and (1.8), $\Delta^i g_n^{(i)} = (-1)^i g_n^{(0)} = (-1)^i$. But (1.12) states that $\Delta^i g_n^{(i)} = (c_1^{(1)} + c_2^{(1)})^i$, implying $c_1^{(1)} + c_2^{(1)} = -1$, and hence, $c_2^{(1)} = -1 - b$, as required.

Proof that $c_2^{(2)} = 1 + b$**.** We equivalently prove that $c_2^{(2)} + c_2^{(1)} = 0$.

By (1.8), $g_n^{(0)} = 1$. By (1.6), $g_1^{(1)} = 1$. Applying (2.4), we derive that $g_2^{(1)} = 0$. By (1.10) and (1.11), $g_2^{(2)} = c_1^{(1)} + c_2^{(2)}$. Finally, by (2.4), $g_3^{(2)} = c_1^{(1)} + c_2^{(2)}$.

Applying (2.8) to $g_3^{(2)}$ and using the above identities we derive that $c_1^{(1)} + c_2^{(2)} = c_1^{(1)} + c_2^{(2)} + c_2^{(1)} + c_2^{(2)}$. The desired result immediately follows.

Sufficiency. We assume (2.7) with $m = 2$ and proceed to derive (2.4).

Using the notation of (2.7), (2.8) states

$$g_n^{(i)} - g_{n-1}^{(i)} = b(g_{n-1}^{(i-1)} - g_{n-2}^{(i-1)}) - g_{n-2}^{(i-1)} + g_{n-2}^{(i-2)} + b g_{n-2}^{(i-2)}. \tag{2.9}$$

By an induction assumption on i, we may assume $g_{n-1}^{(i-1)} - g_{n-2}^{(i-1)} = -g_{n-2}^{(i-2)}$. Plugging this induction assumption into (2.9) and simplifying we have $g_n^{(i)} - g_{n-1}^{(i)} = -g_{n-1}^{(i-1)}$, proving (2.4). $\square$

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3. Proof of the Main Theorem

First, by (1.3), (1.5), and (1.4) we have the following fundamental identity

$$g_n^{(i)} = \sum_{j=1}^{m} \sum_{k=0}^{j} c_j^{(k)} g_{n-j}^{(i-k)}. \tag{3.1}$$

Second, we prove (1.9) which equivalently means that the $\{G_n(x)\}_{n \geq 0}$ are polynomials of degree at most n with no finite poles. This is easy to see, since by (1.5) the degree of $p_i(x)$ is exactly i, $1 \leq i \leq m$, and by (1.6) the degree of $G_i(x)$ is at most i for $0 \leq i \leq m-1$. Therefore, a routine induction using (1.4) shows that the degree of $G_n(x)$ is at most n.

We now prove Theorem 1.1(i), (iii), (iv) using (3.1).

Proof of (1.8)**.** We apply (3.1) with $i = 0$.

By (1.9), for the $g_{n-j}^{(i-k)}$ on the right-hand side of (3.1) to be non-zero, $i - k$ must be non-negative; hence, we infer that $k = 0$. By (1.7), $c_j^{(0)}$ is only non-zero if $j = 1$ and in that case we have $c_j^{(0)} = 1$.

Hence, when $i = 0$, (3.1) reduces to $g_n^{(0)} = c_n^{(0)} g_{n-1}^{(0)} = g_{n-1}^{(0)}$. The proof is completed by an induction argument using (1.6) as a base step.

Proof of (1.11)**.** We apply (3.1) with $i = n$.

By (1.9), for the $g_{n-j}^{(i-k)}$ on the right-hand side of (3.1) to be non-zero, we require $i - k \leq n - j$ or equivalently $k \geq j$. Since the limits in the second summation are from $k = 0$ to $k = j$ we conclude that $k = j$. But then we immediately have that (3.1) reduces to $g_n^{(n)} = \sum_{k=1}^{m} c_k^{(k)} g_{n-k}^{(n-k)}$ proving, using (1.10), (1.11).

Proof of (1.12)**.** The proof is by induction on i.

For a base case, we let $i = 0$ in (1.12). The resulting equation is true by (1.8).

For an induction assumption we assume, for some $i \geq 1$.

$$\Delta^{i-1} g_n^{(i-1)} = \left(\sum_{j=1}^{m} c_j^{(1)} \right)^{i-1}. \tag{3.2}$$

Note, that for $p \geq 0$,

$$\Delta^p g_n^{(p)} = \text{constant, for all sufficiently large } n \text{ implies } \Delta^{p+1} g_n^{(p)} = 0, \text{ for all sufficiently large } n. \tag{3.3}$$

We apply (3.1) with n replaced by $n+1$. By (1.7), we obtain

$$\Delta g_n^{(i)} = g_{n+1}^{(i)} - g_n^{(i)} = c_1^{(1)} g_n^{(i-1)} + \sum_{j=2}^{m} \sum_{k=0}^{j} c_j^{(k)} g_{n+1-j}^{(i-k)}. \tag{3.4}$$

We now apply the operator Δ^{i-1} to both sides of (3.4). We obtain

$$\Delta^i g_n^{(i)} = c_1^{(1)} \Delta^{i-1} g_n^{(i-1)} + \sum_{j=2}^{m} \sum_{k=0}^{j} c_j^{(k)} \Delta^{i-1} g_{n+1-j}^{(i-k)}. \tag{3.5}$$

In simplifying (3.5), we may, by (3.3) discard terms in the second summation with $k > 1$. Similarly, by (1.7), we may discard the $k = 0$ term. In other words, in the second summation we only have left the $k = 1$ term. Hence, (3.5) simplifies to

$$\Delta^i g_n^{(i)} = c_1^{(1)} \Delta^{i-1} g_n^{(i-1)} + \sum_{j=2}^{m} c_j^{(1)} \Delta^{i-1} g_{n+1-j}^{(i-1)}. \tag{3.6}$$

The proof of (1.12) is now completed by substituting (3.2) into (3.6).

Acknowledgement

Acknowledgement to several participants of the West Coast Number Theory conference, 2015, and to several editors of the online encyclopedia of integer sequences (OEIS) for helpful conversations and insights.

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MSC2010: 11B39

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