EXTENDING SOME FIBONACCI–LUCAS RELATIONS

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Abstract. We give a generalization of two recently proven Fibonacci-Lucas identities.

In three recent issues of *The American Mathematical Monthly* (see [1, 2, 3]), two Fibonacci-Lucas relations were demonstrated:

$$2^{m+1}F_{m+1} = \sum_{i=0}^{m} 2^i L_i \qquad \text{and} \qquad 3^{m+1}F_{m+1} = \sum_{i=0}^{m} 3^i L_i + \sum_{i=0}^{m+1} 3^{i-1} F_i.$$

We show these two formulas are part of a family of such identities obtained by replacing 2 and 3, respectively, by any integer $k \geq 1$. Using our formulation, the proof is elementary.

Theorem 1. *For all integers $m \geq 1$ and $k \geq 1$, we have*

$$k^{m+1}F_{m+1} = \sum_{i=0}^{m} k^i L_i + (k-2) \sum_{i=0}^{m+1} k^{i-1} F_i.$$

Proof. It is well-known that $L_i + F_i = F_{i-1} + F_{i+1} + F_i = 2F_{i+1}$ so that, after rearranging sums, we get

$$\begin{aligned} \sum_{i=0}^{m} k^i L_i + (k-2) \sum_{i=0}^{m+1} k^{i-1} F_i &= \sum_{i=0}^{m} k^i (L_i + F_i) + k^{m+1} F_{m+1} - 2 \sum_{i=0}^{m+1} k^{i-1} F_i \\ &= 2 \sum_{i=0}^{m} k^i F_{i+1} + k^{m+1} F_{m+1} - 2 \sum_{i=0}^{m+1} k^{i-1} F_i \\ &= k^{m+1} F_{m+1} \end{aligned}$$

as required (since $F_0 = 0$).

□

References

[1] H. Kwong, *An alternate proof of Sury's Fibonacci–Lucas relation*, Amer. Math. Monthly, **121.6** (2014), 514.

[2] D. Marques, *A new Fibonacci–Lucas relation*, Amer. Math. Monthly, **122.7** (2015), 683.

[3] B. Sury, *A polynomial parent to a Fibonacci–Lucas relation*, Amer. Math. Monthly, **121.3** (2014), 236.

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