SYLVESTER'S THEOREM AND THE NON-INTEGRALITY OF A CERTAIN BINOMIAL SUM

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Abstract. In this note, we show that

$$S(n,r) := \sum_{k=0}^{n} \frac{k}{k+r}\binom{n}{k}$$

is not an integer for any positive integer n and $r \in \{1,2,3,4,5,6\}$ and for $n \le r-1$. This gives a partial answer to a conjecture of [3].

Marcel Chirita [1] asked to show that

$$\sum_{k=0}^{n} \frac{k}{k+1}\binom{n}{k} \notin \mathbb{Z} \tag{1.1}$$

for any integer $n \geq 1$. The first author [3] proved that

$$\sum_{k=0}^{n} \frac{k}{k+r}\binom{n}{k}$$

is not an integer for positive integers n and $r \in \{2,3,4\}$ and asked if the above sum is ever an integer for some positive integers n and r. Plainly, since

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is an integer, the question is equivalent to whether

$$S(n,r) := \sum_{k=0}^{n} \frac{r}{k+r}\binom{n}{k} \tag{1.2}$$

is ever an integer for some positive integers n and r. For $n=1$, we have $S(n,r) = 1 + r/(r+1)$, which is not an integer because it lies inside the interval $(1,2)$; so we may assume that $n \geq 2$. Trying out small values of r we find the formulas:

$$S(n,1) = \frac{2^{n+1}-1}{n+1};$$
$$S(n,2) = (-2)\left(\frac{2^{n+1}-1}{n+1}\right) + 2\left(\frac{2^{n+2}-1}{n+2}\right);$$
$$S(n,3) = 3\left(\frac{2^{n+1}-1}{n+1}\right) - 6\left(\frac{2^{n+2}-1}{n+2}\right) + 3\left(\frac{2^{n+3}-1}{n+3}\right);$$
$$S(n,4) = (-4)\left(\frac{2^{n+1}-1}{n+1}\right) + 12\left(\frac{2^{n+2}-1}{n+2}\right) - 12\left(\frac{2^{n+3}-1}{n+3}\right) + 4\left(\frac{2^{n+4}-1}{n+4}\right);$$
$$S(n,5) = 5\left(\frac{2^{n+1}-1}{n+1}\right) - 20\left(\frac{2^{n+2}-1}{n+2}\right) + 30\left(\frac{2^{n+3}-1}{n+3}\right) - 20\left(\frac{2^{n+4}-1}{n+4}\right) + 5\left(\frac{2^{n+5}-1}{n+5}\right);$$
$$S(n,6) = (-6)\left(\frac{2^{n+1}-1}{n+1}\right) + 30\left(\frac{2^{n+2}-1}{n+2}\right) - 60\left(\frac{2^{n+3}-1}{n+3}\right) + 60\left(\frac{2^{n+4}-1}{n+4}\right) - 30\left(\frac{2^{n+5}-1}{n+5}\right) + 6\left(\frac{2^{n+6}-1}{n+6}\right). \quad (1.3)$$

At this point we recall the well-known fact that n never divides $2^n - 1$ for any $n \geq 2$ (see, for example, problem A14 in [4]).

In particular, $(2^{n+1}-1)/(n+1)$ is not an integer which by the first relation (1.3) deals with the case $r = 1$.

For $r = 2$, one of $n+1$ and $n+2$ is odd. We assume that $n+1$ is odd, since the case when $n+2$ is odd is similar. Then, $2(2^{n+1}-1)/(n+1)$ is a rational number which, in its simplest form, has an odd prime divisor p in its denominator. Since $n+1$ and $n+2$ are coprime, we get that p does not divide $n+2$, so p divides the denominator of $S(n,2)$. Hence, $S(n,2)$ is not an integer.

For $r = 3$, suppose first that $n+1$ is odd. Then so is $n+3$ and one of $n+1$, $n+3$ is not a multiple of 3. Assume $n+1$ is not a multiple of 3, and the case when $n+3$ is not a multiple of 3 can be dealt with similarly. Then $3(2^{n+1}-1)/(n+1)$ is a rational number which, in its simplest form, has a prime factor $p \geq 5$ in its denominator. Clearly, p does not divide either one of $n+2$, $n+3$, so p divides the denominator of $S(n,3)$. Hence, $S(n,3)$ is not an integer. Assume now that $n+1$ is even. In this case, one of $n+1$, $n+3$ is a multiple of 4, and the other is congruent to $2 \pmod 4$, and plainly $n+2$ is odd. The third formula (1.3) now shows easily that $S(n,3)$ is not a 2-adic integer in this case. In fact, its denominator as a rational number is a multiple of 4. This takes care of the case $r = 3$.

For $r = 4$, either $n+1$ or $n+4$ is odd. We assume that $n+1$ is odd since the case when $n+4$ is odd can be dealt with similarly. Then $n+1$ and $n+3$ are both odd and at most one of them is a multiple of 3. Thus, there exists $i \in \{1,3\}$ such that $n+i$ is coprime to 6. Then $c_i(2^{n+i}-1)/(n+i)$ is a rational number, which in its simplest form, has a prime divisor $p \geq 5$ in its denominator. Here, $c_i = 4$ if $i = 1$ and $c_i = 12$ if $i = 3$. This prime p cannot divide $n+j$ for any $j \neq i$, $j \in \{1,2,3,4\}$, therefore p divides the denominator of $S(n,4)$.

For $r = 5$, consider first the case when $n+1$ is odd. Then $n+1$, $n+3$, $n+5$ are all odd. Of these three numbers, at most one is a multiple of 3 and at most one is a multiple of 5. Hence, there is $i \in \{1,3,5\}$ such that $n+i$ is coprime to 30. Then $c_i(2^{n+i}-1)/(n+i)$ is a rational number which, in its simplest form, has a prime factor $p \geq 7$ in its denominator. Here, $c_i = 5, 30, 5$, for $i = 1, 3, 5$, respectively. The prime p cannot divide $n+j$ for any $j \neq i$,

$j \in \{1,2,3,4,5\}$, so $S(n,5)$ is not an integer. Assume now that $n+1$ is even. If $n+1 \equiv 2 \pmod 4$, then $n+3 \equiv 0 \pmod 4$ and $n+5 \equiv 2 \pmod 4$. Hence,

$$5\left(\frac{2^{n+1}-1}{n+1}\right)+30\left(\frac{2^{n+3}-1}{n+3}\right)+5\left(\frac{2^{n+5}-1}{n+5}\right)$$

is a rational number which, in its simplest form, has an even denominator. Since $n+2$, $n+4$ are odd, it follows that $S(n,5)$ is a rational number with an even denominator. Finally, when $n+1 \equiv 0 \pmod 4$, then $n+3 \equiv 2 \pmod 4$ and $n+5 \equiv 0 \pmod 4$. Since $n+1$, $n+5$ are both multiples of 4 whose difference is 4, it follows that one of them is congruent to $4 \pmod 8$ and the other is a multiple of 8. It now follows that the denominator of $S(n,5)$ is even, and in fact, is a multiple of 8. Hence, $S(n,5)$ is not an integer either.

For $r=6$, one of $n+1$ to $n+6$ is odd. We consider only the case when $n+1$ is odd since the case when $n+6$ is odd is similar. Then $n+1$, $n+3$, $n+5$ are all odd and at most one of them is a multiple of 3 and at most one of them is a multiple of 5. Hence, there is $i \in \{1,3,5\}$ such that $n+i$ is coprime to 30, so, in particular, $c_i(2^{n+i}-1)/(n+i)$ is a rational number which, in its simplest form, has a prime factor $p \geq 7$ in its denominator. Here, $c_i = 6, 60, 30$, for $i = 1, 3, 5$, respectively. Clearly, p cannot divide $n+j$ for $j \neq i$, $j \in \{1,2,3,4,5,6\}$, therefore $S(n,6)$ is a rational number whose denominator is a multiple of p.

So far, we reproved the main result from [3] and even proved the cases $r=5$ and $r=6$. In order to extend our argument to cover all r, we need two ingredients:

(i) A general formula of the shape of (1.3) valid for n and r;
(ii) A statement about prime factors of consecutive integers, namely that under some mild hypothesis, out of every r consecutive integers there is one of them divisible by a prime larger than r.

The next statement takes care of (i) and, in particular, justifies formulas (1.3).

Lemma 1. *We have*

$$S(n,r)=\sum_{j=0}^{r-1}(-1)^{r-1-j}r\binom{r-1}{j}\left(\frac{2^{n+j+1}-1}{n+j+1}\right). \tag{1.4}$$

Proof.

$$\begin{aligned}
S(n,r) &= r\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+r} = r\sum_{k=0}^{n}\binom{n}{k}\int_0^1 x^{k+r-1}dx \\
&= r\int_0^1\left(\sum_{k=0}^{n}\binom{n}{k}x^{k+r-1}\right)dx = r\int_0^1\left(\sum_{k=0}^{n}\binom{n}{k}x^{k}\right)x^{r-1}dx \\
&= r\int_0^1 (1+x)^n x^{r-1}dx = r\int_0^1(1+x)^n(1+x-1)^{r-1}dx \\
&= r\int_0^1 (1+x)^n\left(\sum_{j=0}^{r-1}(-1)^{r-1-j}\binom{r-1}{j}(1+x)^j\right)dx \\
&= \int_0^1\left(\sum_{j=0}^{r-1}(-1)^{r-1-j}r\binom{r-1}{j}(1+x)^{n+j}\right)dx
\end{aligned}$$

$$= \sum_{j=0}^{r-1} (-1)^{r-1-j} r \binom{r-1}{j} \int_0^1 (1+x)^{n+j} dx$$
$$= \sum_{j=0}^{r-1} (-1)^{r-1-j} r \binom{r-1}{j} \left(\frac{2^{n+j+1} - 1}{n+j+1} \right).$$

□

For (ii), let us recall Sylvester's extension of Bertrand's postulate (see [2]).

Theorem 2. *If $n \geq r \geq 2$, then one of the numbers $n+1$, $n+2, \ldots, n+r$ is divisible by a prime larger than r.*

However, Sylvester's Theorem is not enough to prove that $S(n,r)$ is not an integer for any n and r, even when $n \geq r$, because although we infer that there exists $i \in \{1, 2, \ldots, r\}$ such that $p \mid n+i$ for some prime $p > r$, and $n+i$ does not divide $2^{n+i} - 1$, it is still possible that $c_i(2^{n+i} - 1)/(n+i)$ is a rational number whose denominator is *not* divisible by p, and therefore we cannot infer that p divides the denominator of $S(n,r)$. However, Sylvester's Theorem is enough to deal with the case $n \leq r-1$. Namely, in this case, we work directly with the original representation of (1.2), which is

$$S(n,r) = 1 + \sum_{j=1}^{n} \frac{r}{r+j} \binom{n}{j}.$$

If $r+1 > n$, then, again by Sylvester's Theorem, one of the numbers $r+1, r+2, \ldots, r+n$ is divisible by a prime $p > n$. Such a prime does not divide $\binom{n}{j}$ for any $j \in \{1, \ldots, n\}$, and does not divide r either (otherwise, it divides both r and $r+j$ for some $j \in \{1, \ldots, n\}$, so it divides their difference, which is a number less than or equal to n, a contradiction). So, it remains to deal with $r = n+1$. In this case, we apply Bertrand's postulate, to conclude that there is a prime $p \in ((n+1), 2n+1]$. This prime divides neither $n+1$ nor $\binom{n}{j}$ for $j \in \{1, \ldots, n\}$, so p divides the denominator of $S(n, n+1)$.

To summarize, in this note we proved, in addition to formula (1.4), the following partial results towards the conjecture that $S(n,r)$ is not an integer for any positive integers n and r:

Theorem 3.

(1) *$S(n,r)$ is not an integer for any $r \in \{1, 2, 3, 4, 5, 6\}$ and $n \geq 2$;*
(2) *$S(n,r)$ is not an integer for $1 \leq n \leq r-1$.*

Acknowledgements

We thank the anonymous referee for comments which improved the quality of our paper. We also thank Tewodoros Amdeberhan, José Hernández Santiago, and Shanta Laishram for useful correspondence on the problem treated in this paper.

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MSC2010: 11B65

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