MORE NEW ALGEBRAIC IDENTITIES AND THE FIBONACCI SUMMATIONS DERIVED FROM THEM

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Abstract. In this paper, we introduce certain algebraic identities that we believe are new. For each of these algebraic identities, one side telescopes when summed. A link to the Fibonacci/Lucas numbers then facilitates the derivation of closed forms for reciprocal series that involve the Fibonacci/Lucas numbers. The term that defines the denominator of each summand contains squares of Fibonacci related numbers, with subscripts in arithmetic progression.

1. Introduction

The Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$$

With $\alpha = (1+\sqrt{5})/2$, the Binet (closed) forms for F_n and L_n are

$$F_n = \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{5},$$
$$L_n = \alpha^n + (-1)^n\alpha^{-n},$$

and these closed forms are valid for all integers n.

In the recent paper [3], we present closed forms for Fibonacci related reciprocal sums. For instance, we give closed forms, both finite and infinite, for summands of the form

$$\frac{L_{2ai+b}}{(F_{2ai+b}+c)^2} \text{ and } \frac{F_{2ai+b}}{(L_{2ai+b}+c)^2},$$

for certain values of the constants a, b, and c. We first establish algebraic identities that telescope when summed, and then transform the telescoping sums derived from these algebraic identities into Fibonacci/Lucas numbers.

One of the algebraic identities that appears in [3] is

$$\frac{\left(t^a - t^{-a}\right)\left(t^{2an+b} - t^{-2an-b}\right)}{\left(t^{2an+b} + t^{-2an-b} + t^a + t^{-a}\right)^2} = \frac{t^{2an+b-a}}{\left(1+t^{2an+b-a}\right)^2} - \frac{t^{2an+b+a}}{\left(1+t^{2an+b+a}\right)^2},$$

in which $t > 1$ is a real number, and n, a, and b are integers. Interestingly, the algebraic identity in question, together with the telescoping sum derived from it, have nothing to do with Fibonacci/Lucas numbers. The Fibonacci connection comes about when we replace t by α.

In this paper, we continue the line of research described above. For instance, with the use of algebraic identities that telescope when summed, we present closed forms for sums (both finite and infinite) associated with the summands

$$\frac{1}{F_{2ai+b}^2 + c} \text{ and } \frac{F_{2ai+b}}{F_{2ai+b}^2 + c},$$

for certain values of the constants a, b, and c. We also consider other summands, including summands that alternate with the running variable i.

2. Finite Sums I

In this section, we present the first of our results on finite sums. Throughout the paper, n, a, b, and s are taken to be integers, and henceforth we do not restate this. Precisely which integers these parameters represent will be made clear in each situation. In Lemma 2.1, the condition $a \neq b$ is imposed to exclude the possibility of vanishing denominators when $n = 0$.

We now present our first algebraic identity. This algebraic identity, that we believe to be new, is the source of all the results in this section.

Lemma 2.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, with $a \neq b$. Then*

$$\frac{(t^a - t^{-a})\left(t^{2an+b} + t^{-2an-b}\right)}{(t^{2an+b} + t^{-2an-b})^2 - (t^a + t^{-a})^2} = \frac{1}{t^{a(2n-1)+b} - t^{-a(2n-1)-b}} - \frac{1}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \tag{2.1}$$

Proof. The product of the denominators of the algebraic fractions on the right side of (2.1) equals the denominator of the algebraic fraction on the left side of (2.1). The proof of (2.1) now follows by simple algebra. □

The main results in this section, which we present in Theorem 2.2, follow from Lemma 2.1.

Theorem 2.2. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$L_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{F_{2ai+b}^2 - F_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{ if } a \neq b \text{ are both odd,} \tag{2.2}$$

$$5F_a \sum_{i=0}^{n} \frac{L_{2ai+b}}{L_{2ai+b}^2 - L_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{ if } a \neq b \text{ are both even,} \tag{2.3}$$

$$5F_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{5F_{2ai+b}^2 - L_a^2} = \frac{1}{L_{b-a}} - \frac{1}{L_{a(2n+1)+b}}, \text{ if } a \text{ is even and } b \text{ is odd,} \tag{2.4}$$

$$L_a \sum_{i=0}^{n} \frac{L_{2ai+b}}{L_{2ai+b}^2 - 5F_a^2} = \frac{1}{L_{b-a}} - \frac{1}{L_{a(2n+1)+b}}, \text{ if } a \text{ is odd and } b \text{ is even.} \tag{2.5}$$

Proof. By the telescoping effect, it follows from (2.1) that

$$\sum_{i=0}^{n} \frac{(t^a - t^{-a})\left(t^{2ai+b} + t^{-2ai-b}\right)}{(t^{2ai+b} + t^{-2ai-b})^2 - (t^a + t^{-a})^2} = \frac{1}{t^{b-a} - t^{-(b-a)}} - \frac{1}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \tag{2.6}$$

In (2.6), let $t = \alpha$. There are four cases that arise from the possible parities of a and b. For each of these cases, we use the Binet forms to transform (2.6) into Fibonacci/Lucas numbers. The four different sums that arise are those given in Theorem 2.2. □

Four infinite sums follow immediately from Theorem 2.2, and we leave these to the reader to write down.

Before proceeding, we issue a note of caution. If, in the denominator of the fraction on the left side of (2.1), we replace $t^{2an+b} + t^{-2an-b}$ by $t^{2an+b} - t^{-2an-b}$, and $t^a + t^{-a}$ by $t^a - t^{-a}$, we obtain an identity similar to (2.1). From this identity, four finite sums similar to those in Theorem 2.2 can be written down. However, these four sums are merely equivalent versions of (2.2)–(2.5). This can be explained with the use of the well known identity

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{2.7}$$

From (2.7) it is seen that

$$L_m^2 - L_n^2 = 5\left(F_m^2 - F_n^2\right) \tag{2.8}$$

if m and n have the same parity, while

$$L_m^2 - 5F_n^2 = 5F_m^2 - L_n^2 \tag{2.9}$$

if m and n have different parities.

Identity (2.8) shows that equivalent versions of (2.2) and (2.3) are, respectively,

$$5L_a \sum_{i=0}^{n} \frac{F_{2ai+b}}{L_{2ai+b}^2 - L_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{if } a \neq b \text{ are both odd,}$$

$$F_a \sum_{i=0}^{n} \frac{L_{2ai+b}}{F_{2ai+b}^2 - F_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{if } a \neq b \text{ are both even.}$$

With the use of (2.9), it is easy to write down equivalent versions of (2.4) and (2.5).

The observations that we have made in the previous two paragraphs apply to *all* the sums that we present in this paper.

3. Finite Sums II

In this section, we give closed forms for four alternating finite sums. These sums are derived from the algebraic identity contained in Lemma 3.1. We do not present a proof Lemma 3.1 since its proof is similar to that of Lemma 2.1.

Lemma 3.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, with $a \neq b$. Then*

$$\frac{\left(t^a + t^{-a}\right)\left(t^{2an+b} - t^{-2an-b}\right)}{\left(t^{2an+b} - t^{-2an-b}\right)^2 - \left(t^a - t^{-a}\right)^2} = \frac{1}{t^{a(2n-1)+b} - t^{-a(2n-1)-b}} + \frac{1}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \tag{3.1}$$

Theorem 3.2, the main result of this section, gives the alternating counterparts of (2.2)-(2.5).

Theorem 3.2. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$L_a \sum_{i=0}^{n} \frac{(-1)^i F_{2ai+b}}{F_{2ai+b}^2 - F_a^2} = \frac{1}{F_{b-a}} + \frac{(-1)^n}{F_{a(2n+1)+b}}, \textit{if } a \neq b \textit{ are both even,} \tag{3.2}$$

$$5F_a \sum_{i=0}^{n} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 - L_a^2} = \frac{1}{F_{b-a}} + \frac{(-1)^n}{F_{a(2n+1)+b}}, \textit{if } a \neq b \textit{ are both odd,} \tag{3.3}$$

$$5F_a \sum_{i=0}^{n} \frac{(-1)^i F_{2ai+b}}{5F_{2ai+b}^2 - L_a^2} = \frac{1}{L_{b-a}} + \frac{(-1)^n}{L_{a(2n+1)+b}}, \textit{if } a \textit{ is odd and } b \textit{ is even,} \tag{3.4}$$

$$L_a \sum_{i=0}^{n} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 - 5F_a^2} = \frac{1}{L_{b-a}} + \frac{(-1)^n}{L_{a(2n+1)+b}}, \textit{if } a \textit{ is even and } b \textit{ is odd.} \tag{3.5}$$

Proof. In order for the right side of (3.1) to telescope when summed, alternating signs are needed. We then have

$$\begin{aligned}
&\sum_{i=0}^{n} \frac{(-1)^i \left(t^a + t^{-a}\right)\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 - \left(t^a - t^{-a}\right)^2} \\
&= \sum_{i=0}^{n} (-1)^i \left(\frac{1}{t^{a(2i-1)+b} - t^{-a(2i-1)-b}} + \frac{1}{t^{a(2i+1)+b} - t^{-a(2i+1)-b}} \right) \\
&= \frac{1}{t^{b-a} - t^{-(b-a)}} + \frac{(-1)^n}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \qquad (3.6)
\end{aligned}$$

In (3.6), put $t = \alpha$. The proof of Theorem 3.2 now follows along the same lines as the proof of Theorem 2.2. □

Four infinite sums follow from (3.2)–(3.5), and we leave the task of writing these down to the interested reader.

4. Finite Sums III

In this section, we present closed forms for two finite sums. In each of these sums, the numerator of the summand is constant, while the denominator involves certain squared terms with subscripts in arithmetic progression. The sums that we present are derived from the algebraic identity contained in Lemma 4.1. We do not present a proof Lemma 4.1 since its proof is similar to that of Lemma 2.1.

Lemma 4.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, with $a \neq b$. Then*

$$\frac{t^{2a} - t^{-2a}}{\left(t^{2an+b} - t^{-2an-b}\right)^2 - \left(t^a - t^{-a}\right)^2} = \frac{t^{a(2n-1)+b}}{t^{a(2n-1)+b} - t^{-a(2n-1)-b}} - \frac{t^{a(2n+1)+b}}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \qquad (4.1)$$

Theorem 4.2. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$2F_{2a} \sum_{i=0}^{n} \frac{1}{F_{2ai+b}^2 - F_a^2} = \frac{L_{b-a}}{F_{b-a}} - \frac{L_{a(2n+1)+b}}{F_{a(2n+1)+b}}, \text{ if } a \neq b \text{ have the same parity,} \qquad (4.2)$$

$$2F_{2a} \sum_{i=0}^{n} \frac{1}{5F_{2ai+b}^2 - L_a^2} = \frac{F_{b-a}}{L_{b-a}} - \frac{F_{a(2n+1)+b}}{L_{a(2n+1)+b}}, \text{ if } a \text{ and } b \text{ have different parities.} \qquad (4.3)$$

Proof. Upon summing both sides of (4.1), we see that the right side telescopes to yield

$$\sum_{i=0}^{n} \frac{t^{2a} - t^{-2a}}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 - \left(t^a - t^{-a}\right)^2} = \frac{t^{b-a}}{t^{b-a} - t^{-(b-a)}} - \frac{t^{a(2n+1)+b}}{t^{a(2n+1)+b} - t^{-a(2n+1)-b}}. \qquad (4.4)$$

The two identities $\alpha^n = \alpha F_n + F_{n-1}$, and $L_n = F_{n+1} + F_{n-1}$, are well-known. With the use of these two identities, it is shown that

$$\frac{\alpha^n}{F_n} = \frac{L_n}{2F_n} + \frac{\sqrt{5}}{2}, \text{ for all integers } n \neq 0, \qquad (4.5)$$

$$\frac{\alpha^n}{L_n} = \frac{\sqrt{5}F_n}{2L_n} + \frac{1}{2}, \text{ for all integers } n. \qquad (4.6)$$

With $t = \alpha$, and under the assumption that a and b have the same parities, we use the Binet forms, together with (4.5), to transform (4.4) into (4.2). Similarly, under the assumption that a and b have different parities, we make use of (4.6) to transform (4.4) into (4.3). This completes the proof of Theorem 4.2. □

5. Infinite Sums I

The algebraic identities that occur in Lemmas 2.1, 3.1, and 4.1 produce closed forms for certain finite sums. Beginning in this section, we give closed forms for infinite sums that are not readily obtained from the finite sums in Theorems 2.2, 3.2, and 4.2. The infinite sums in question arise from algebraic identities that are more general in nature than those in Sections 2-4. The first of these algebraic identities is stated in Lemma 5.2, the proof of which is similar to the proof of Lemma 2.1. Before stating Lemma 5.2, we specify a condition that excludes the possibility of vanishing denominators.

Condition 5.1. *Let $a \neq 0$, b, and s be integers. Then we say that a, b, and s satisfy Condition 5.1 if $2a|(as-b)$ implies that $as-b<0$.*

Lemma 5.2. *Let $t>1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$\begin{aligned} &\frac{\left(t^{as}-t^{-as}\right)\left(t^{2an+b}+t^{-2an-b}\right)}{\left(t^{2an+b}+t^{-2an-b}\right)^2-\left(t^{as}+t^{-as}\right)^2} \\ &= \frac{1}{t^{a(2n-s)+b}-t^{-a(2n-s)-b}} - \frac{1}{t^{a(2n+s)+b}-t^{-a(2n+s)-b}}. \end{aligned} \tag{5.1}$$

As a consequence of Lemma 5.2, we have

Lemma 5.3. *Let $t>1$ be a real number. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$\sum_{i=0}^{\infty} \frac{\left(t^{as}-t^{-as}\right)\left(t^{2ai+b}+t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}\right)^2-\left(t^{as}+t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{1}{t^{a(2i-s)+b}-t^{-a(2i-s)-b}}. \tag{5.2}$$

Proof. Consider (5.1) for $n>s$. Then, by the telescoping effect,

$$\begin{aligned} &\sum_{i=0}^{n} \frac{\left(t^{as}-t^{-as}\right)\left(t^{2ai+b}+t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}\right)^2-\left(t^{as}+t^{-as}\right)^2} \\ &= \sum_{i=0}^{s-1} \frac{1}{t^{a(2i-s)+b}-t^{-a(2i-s)-b}} - \sum_{i=n-s+1}^{n} \frac{1}{t^{a(2i+s)+b}-t^{-a(2i+s)-b}}. \end{aligned} \tag{5.3}$$

Upon letting $n \to \infty$ in (5.3) we obtain (5.2). □

In (5.2), let $t=\alpha$. Then, considering the various parities of a, b, and s, we make use of the Binet forms to transform (5.2) into Fibonacci/Lucas numbers. We record the various outcomes in the theorem that follows, which is the main result in this section.

Theorem 5.4. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$5F_{as} \sum_{i=0}^{\infty} \frac{L_{2ai+b}}{L_{2ai+b}^2 - L_{as}^2} = \sum_{i=0}^{s-1} \frac{1}{F_{a(2i-s)+b}}, \text{ if } b \text{ is even and } a \text{ or } s \text{ is even,} \tag{5.4}$$

$$L_{as} \sum_{i=0}^{\infty} \frac{L_{2ai+b}}{L_{2ai+b}^2 - 5F_{as}^2} = \sum_{i=0}^{s-1} \frac{1}{L_{a(2i-s)+b}}, \text{ if } b \text{ is even and } a \text{ and } s \text{ are odd,} \tag{5.5}$$

$$5F_{as} \sum_{i=0}^{\infty} \frac{F_{2ai+b}}{5F_{2ai+b}^2 - L_{as}^2} = \sum_{i=0}^{s-1} \frac{1}{L_{a(2i-s)+b}}, \text{ if } b \text{ is odd and } a \text{ or } s \text{ is even,} \tag{5.6}$$

$$L_{as} \sum_{i=0}^{\infty} \frac{F_{2ai+b}}{F_{2ai+b}^2 - F_{as}^2} = \sum_{i=0}^{s-1} \frac{1}{F_{a(2i-s)+b}}, \text{ if } b \text{ is odd and } a \text{ and } s \text{ are odd.} \tag{5.7}$$

The four infinite sums that arise by letting $n \to \infty$ in (2.2)–(2.5) are obtained by putting $s = 1$ in (5.4)–(5.7).

The fact that, for n odd, $L_{-n} = -L_n$, allows us to write down two interesting special cases of Theorem 5.4. These are

$$\sum_{i=0}^{\infty} \frac{L_{2i}}{L_{2i}^2 - 5F_s^2} = \frac{-1}{L_s^2}, \text{ if } s \geq 1 \text{ is odd,} \tag{5.8}$$

$$\sum_{i=0}^{\infty} \frac{F_{2i+1}}{5F_{2i+1}^2 - L_s^2} = 0, \text{ if } s \geq 2 \text{ is even.} \tag{5.9}$$

Here (5.8) arises from (5.5). For instance, with $s = 1$, (5.8) becomes

$$\sum_{i=0}^{\infty} \frac{L_{2i}}{L_{2i}^2 - 5} = -1.$$

The sum (5.9) is a special case of (5.6). From (5.9) we have

$$\sum_{i=0}^{\infty} \frac{F_{2i+1}}{5F_{2i+1}^2 - 9} = \sum_{i=0}^{\infty} \frac{F_{2i+1}}{5F_{2i+1}^2 - 49} = \sum_{i=0}^{\infty} \frac{F_{2i+1}}{5F_{2i+1}^2 - 324} = \cdots 0.$$

6. Infinite Sums II

In this section, our main result, Theorem 6.3, gives closed forms for infinite sums where, in each case, the summand alternates in sign. We require a lemma that we state without proof since its proof is similar to the proofs of previous lemmas.

Lemma 6.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$\frac{\left(t^{as} + t^{-as}\right)\left(t^{2an+b} - t^{-2an-b}\right)}{\left(t^{2an+b} - t^{-2an-b}\right)^2 - \left(t^{as} - t^{-as}\right)^2} = \frac{1}{t^{a(2n-s)+b} - t^{-a(2n-s)-b}} + \frac{1}{t^{a(2n+s)+b} - t^{-a(2n+s)-b}}. \tag{6.1}$$

As a consequence of Lemma 6.1, we have the following lemma.

Lemma 6.2. *Let $t > 1$ be a real number. Let $s \geq 1$ be odd, and assume that $a \geq 1$, $b \geq 0$, and s satisfy Condition 5.1. Then*

$$\sum_{i=0}^{\infty} \frac{(-1)^i \left(t^{as} + t^{-as}\right)\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 - \left(t^{as} - t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{(-1)^i}{t^{a(2i-s)+b} - t^{-a(2i-s)-b}}. \tag{6.2}$$

Proof. Consider (6.1) for $n > s$. Alternating signs are required for the right side of (6.1) to telescope when summed. Furthermore, s must be odd. We then have

$$\sum_{i=0}^{n} \frac{(-1)^i \left(t^{as} + t^{-as}\right)\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 - \left(t^{as} - t^{-as}\right)^2}$$
$$= \sum_{i=0}^{s-1} \frac{(-1)^i}{t^{a(2i-s)+b} - t^{-a(2i-s)-b}} + \sum_{i=n-s+1}^{n} \frac{(-1)^i}{t^{a(2i+s)+b} - t^{-a(2i+s)-b}}. \tag{6.3}$$

Upon letting $n \to \infty$ in (6.3) we obtain (6.2). □

Our next theorem, the proof of which is similar to the proof of Theorem 5.4, is an immediate consequence of (6.2).

Theorem 6.3. *Let $s \geq 1$ be odd, and assume that $a \geq 1$, $b \geq 0$, and s satisfy Condition 5.1. Then*

$$L_{as} \sum_{i=0}^{\infty} \frac{(-1)^i F_{2ai+b}}{F_{2ai+b}^2 - F_{as}^2} = \sum_{i=0}^{s-1} \frac{(-1)^i}{F_{a(2i-s)+b}}, \text{ if } a \text{ and } b \text{ are both even}, \tag{6.4}$$

$$5F_{as} \sum_{i=0}^{\infty} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 - L_{as}^2} = \sum_{i=0}^{s-1} \frac{(-1)^i}{F_{a(2i-s)+b}}, \text{ if } a \text{ and } b \text{ are both odd}, \tag{6.5}$$

$$5F_{as} \sum_{i=0}^{\infty} \frac{(-1)^i F_{2ai+b}}{5F_{2ai+b}^2 - L_{as}^2} = \sum_{i=0}^{s-1} \frac{(-1)^i}{L_{a(2i-s)+b}}, \text{ if } a \text{ is odd and } b \text{ is even}, \tag{6.6}$$

$$L_{as} \sum_{i=0}^{\infty} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 - 5F_{as}^2} = \sum_{i=0}^{s-1} \frac{(-1)^i}{L_{a(2i-s)+b}}, \text{ if } a \text{ is even and } b \text{ is odd}. \tag{6.7}$$

We mention only one consequence of Theorem 6.3. From (6.6), it follows that

$$\sum_{i=0}^{\infty} \frac{(-1)^i F_{2i}}{5F_{2i}^2 - L_s^2} = \frac{1}{5F_s} \sum_{i=0}^{s-1} \frac{(-1)^i}{L_{2i-s}}, \tag{6.8}$$

where $s \geq 1$ must be odd. Thus, for $s = 1$ and $s = 3$, (6.8) becomes, respectively,

$$\sum_{i=0}^{\infty} \frac{(-1)^i F_{2i}}{5F_{2i}^2 - 1} = -\frac{1}{5},$$
$$\sum_{i=0}^{\infty} \frac{(-1)^i F_{2i}}{5F_{2i}^2 - 16} = \frac{7}{40}.$$

7. Infinite Sums III

In this section, we give closed forms for two infinite sums where, in each case, the numerator of the summand is unity. To this end, we require the two lemmas that follow, the first of which we state without proof.

Lemma 7.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$\frac{t^{2as} - t^{-2as}}{(t^{2an+b} - t^{-2an-b})^2 - (t^{as} - t^{-as})^2} = \frac{t^{a(2n-s)+b}}{t^{a(2n-s)+b} - t^{-a(2n-s)-b}} - \frac{t^{a(2n+s)+b}}{t^{a(2n+s)+b} - t^{-a(2n+s)-b}}. \tag{7.1}$$

Lemma 7.2. *Let $t > 1$ be a real number. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$\sum_{i=0}^{\infty} \frac{t^{2as} - t^{-2as}}{(t^{2ai+b} - t^{-2ai-b})^2 - (t^{as} - t^{-as})^2} = \sum_{i=0}^{s-1} \frac{t^{a(2i-s)+b}}{t^{a(2i-s)+b} - t^{-a(2i-s)-b}} - s. \tag{7.2}$$

Proof. Consider (7.1) for $n > s$. Then, by the telescoping effect,

$$\sum_{i=0}^{n} \frac{t^{2as} - t^{-2as}}{(t^{2ai+b} - t^{-2ai-b})^2 - (t^{as} - t^{-as})^2}$$
$$= \sum_{i=0}^{s-1} \frac{t^{a(2i-s)+b}}{t^{a(2i-s)+b} - t^{-a(2i-s)-b}} - \sum_{i=n-s+1}^{n} \frac{t^{a(2i+s)+b}}{t^{a(2i+s)+b} - t^{-a(2i+s)-b}}. \tag{7.3}$$

Upon letting $n \to \infty$ in (7.3) we obtain (7.2). □

Letting $t = \alpha$ in (7.2) gives

$$\sum_{i=0}^{\infty} \frac{\alpha^{2as} - \alpha^{-2as}}{(\alpha^{2ai+b} - \alpha^{-2ai-b})^2 - (\alpha^{as} - \alpha^{-as})^2} = \sum_{i=0}^{s-1} \frac{\alpha^{a(2i-s)+b}}{\alpha^{a(2i-s)+b} - \alpha^{-a(2i-s)-b}} - s. \tag{7.4}$$

Based on our discussion at the end of Section 2, the closed forms for only two distinct infinite sums flow from (7.4). These infinite sums are presented in the theorem that follows.

Theorem 7.3. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$ satisfy Condition 5.1. Then*

$$2F_{2as} \sum_{i=0}^{\infty} \frac{1}{F_{2ai+b}^2 - F_{as}^2} = \sum_{i=0}^{s-1} \frac{L_{a(2i-s)+b}}{F_{a(2i-s)+b}} - s\sqrt{5}, \text{ if } as \text{ and } b \text{ have the same parity,} \tag{7.5}$$

$$2F_{2as} \sum_{i=0}^{\infty} \frac{1}{5F_{2ai+b}^2 - L_{as}^2} = \sum_{i=0}^{s-1} \frac{F_{a(2i-s)+b}}{L_{a(2i-s)+b}} - \frac{s}{\sqrt{5}}, \text{ if } as \text{ and } b \text{ have different parities.} \tag{7.6}$$

Proof. Under the assumption that as and b have the same parity, we use the Binet forms, together with (4.5), to transform (7.4) into (7.5). Similarly, under the assumption that as and b have different parities, we make use of the Binet forms, together with (4.6), to transform (7.4) into (7.6). □

In (7.6), set $(a, b) = (1, 0)$. Then

$$2F_{2s} \sum_{i=0}^{\infty} \frac{1}{5F_{2i}^2 - L_s^2} = \sum_{i=0}^{s-1} \frac{F_{2i-s}}{L_{2i-s}} - \frac{s}{\sqrt{5}}, \text{if } s \geq 1 \text{ is odd.} \tag{7.7}$$

With $s = 1$, (7.7) becomes

$$\sum_{i=0}^{\infty} \frac{1}{5F_{2i}^2 - 1} = -\frac{5 + \sqrt{5}}{10}.$$

Again, in (7.6) set $(a, b) = (1, 1)$. Then

$$2F_{2s}\sum_{i=0}^{\infty}\frac{1}{5F_{2i+1}^2 - L_s^2} = \sum_{i=0}^{s-1}\frac{F_{2i+1-s}}{L_{2i+1-s}} - \frac{s}{\sqrt{5}}, \text{if } s \geq 2 \text{ is even.} \tag{7.8}$$

With $s = 2$, (7.8) becomes

$$\sum_{i=0}^{\infty}\frac{1}{5F_{2i+1}^2 - 9} = -\frac{\sqrt{5}}{15}.$$

8. Analogues of Theorem 2.2, Theorem 3.2, and Theorem 4.2

In this section, we present three algebraic identities together with the finite sums that they produce. Indeed, the finite sums produced are analogous to the finite sums given in Theorems 2.2, 3.2, and 4.2. Once again, identities (2.8) and (2.9), with terms transposed, can be used to prevent the duplication of identities. In this section, and the next, the condition $t > 1$ is enough to prevent the occurrence of vanishing denominators. Since our methods are now clear, we state our results without proof, and invite the interested reader to supply the details.

Lemma 8.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$\frac{(t^a - t^{-a})\left(t^{2an+b} - t^{-2an-b}\right)}{(t^{2an+b} - t^{-2an-b})^2 + (t^a + t^{-a})^2} = \frac{1}{t^{a(2n-1)+b} + t^{-a(2n-1)-b}} - \frac{1}{t^{a(2n+1)+b} + t^{-a(2n+1)-b}}. \tag{8.1}$$

Lemma 8.1 yields the following theorem.

Theorem 8.2. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$L_a\sum_{i=0}^{n}\frac{L_{2ai+b}}{L_{2ai+b}^2 + 5F_a^2} = \frac{1}{L_{b-a}} - \frac{1}{L_{a(2n+1)+b}}, \text{ if } a \text{ and } b \text{ are both odd,} \tag{8.2}$$

$$5F_a\sum_{i=0}^{n}\frac{F_{2ai+b}}{5F_{2ai+b}^2 + L_a^2} = \frac{1}{L_{b-a}} - \frac{1}{L_{a(2n+1)+b}}, \text{ if } a \text{ and } b \text{ are both even,} \tag{8.3}$$

$$5F_a\sum_{i=0}^{n}\frac{L_{2ai+b}}{L_{2ai+b}^2 + L_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{ if } a \text{ is even and } b \text{ is odd,} \tag{8.4}$$

$$L_a\sum_{i=0}^{n}\frac{F_{2ai+b}}{F_{2ai+b}^2 + F_a^2} = \frac{1}{F_{b-a}} - \frac{1}{F_{a(2n+1)+b}}, \text{ if } a \text{ is odd and } b \text{ is even.} \tag{8.5}$$

Lemma 8.3. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$\frac{(t^a + t^{-a})\left(t^{2an+b} + t^{-2an-b}\right)}{(t^{2an+b} + t^{-2an-b})^2 + (t^a - t^{-a})^2} = \frac{1}{t^{a(2n-1)+b} + t^{-a(2n-1)-b}} + \frac{1}{t^{a(2n+1)+b} + t^{-a(2n+1)-b}}. \tag{8.6}$$

Upon multiplying (8.6) by $(-1)^i$, we see that the right side telescopes when summed. The outcome leads to the following theorem.

Theorem 8.4. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$L_a \sum_{i=0}^{n} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 + 5F_a^2} = \frac{1}{L_{b-a}} + \frac{(-1)^n}{L_{a(2n+1)+b}}, \text{ if } a \text{ and } b \text{ are both even,} \tag{8.7}$$

$$5F_a \sum_{i=0}^{n} \frac{(-1)^i F_{2ai+b}}{5F_{2ai+b}^2 + L_a^2} = \frac{1}{L_{b-a}} + \frac{(-1)^n}{L_{a(2n+1)+b}}, \text{ if } a \text{ and } b \text{ are both odd,} \tag{8.8}$$

$$5F_a \sum_{i=0}^{n} \frac{(-1)^i L_{2ai+b}}{L_{2ai+b}^2 + L_a^2} = \frac{1}{F_{b-a}} + \frac{(-1)^n}{F_{a(2n+1)+b}}, \text{ if } a \text{ is odd and } b \text{ is even,} \tag{8.9}$$

$$L_a \sum_{i=0}^{n} \frac{(-1)^i F_{2ai+b}}{F_{2ai+b}^2 + F_a^2} = \frac{1}{F_{b-a}} + \frac{(-1)^n}{F_{a(2n+1)+b}}, \text{ if } a \text{ is even and } b \text{ is odd.} \tag{8.10}$$

Lemma 8.5. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$\frac{t^{2a} - t^{-2a}}{\left(t^{2an+b} - t^{-2an-b}\right)^2 + \left(t^a + t^{-a}\right)^2} = \frac{t^{-a(2n-1)-b}}{t^{a(2n-1)+b} + t^{-a(2n-1)-b}} - \frac{t^{-a(2n+1)-b}}{t^{a(2n+1)+b} + t^{-a(2n+1)-b}}. \tag{8.11}$$

Finally, for this section, keeping in mind that $F_{-n} = (-1)^{n+1}F_n$, and $L_{-n} = (-1)^n L_n$, we make use of (4.5) and (4.6), together with Lemma 8.5, to obtain the following theorem.

Theorem 8.6. *Let $n \geq 0$. Let $a \geq 1$, $b \geq 0$. Then*

$$2F_{2a} \sum_{i=0}^{n} \frac{1}{5F_{2ai+b}^2 + L_a^2} = \frac{F_{a(2n+1)+b}}{L_{a(2n+1)+b}} - \frac{F_{b-a}}{L_{b-a}}, \text{ if } a \text{ and } b \text{ have the same parity,} \tag{8.12}$$

$$2F_{2a} \sum_{i=0}^{n} \frac{1}{F_{2ai+b}^2 + F_a^2} = \frac{L_{a(2n+1)+b}}{F_{a(2n+1)+b}} - \frac{L_{b-a}}{F_{b-a}}, \text{ if } a \text{ and } b \text{ have different parities.} \tag{8.13}$$

9. Analogues of Theorem 5.4, Theorem 6.3, and Theorem 7.3

In this section, we present three theorems that are analogous to Theorems 5.4, 6.3, and 7.3. The results stated in these three theorems are achieved via the introduction of three algebraic identities that can be considered as counterparts to those proved in Lemmas 5.2, 6.1, and 7.1. As in the previous section, since our methods are now clear, we state the results of this section with minimal commentary.

Lemma 9.1. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\frac{\left(t^{as} - t^{-as}\right)\left(t^{2an+b} - t^{-2an-b}\right)}{\left(t^{2an+b} - t^{-2an-b}\right)^2 + \left(t^{as} + t^{-as}\right)^2} = \frac{1}{t^{a(2n-s)+b} + t^{-a(2n-s)-b}} - \frac{1}{t^{a(2n+s)+b} + t^{-a(2n+s)-b}}. \tag{9.1}$$

Lemma 9.1 yields

$$\sum_{i=0}^{\infty} \frac{\left(t^{as} - t^{-as}\right)\left(t^{2ai+b} - t^{-2ai-b}\right)}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 + \left(t^{as} + t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{1}{t^{a(2i-s)+b} + t^{-a(2i-s)-b}}, \tag{9.2}$$

from which Theorem 9.2 follows.

Theorem 9.2. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$5F_{as}\sum_{i=0}^{\infty}\frac{F_{2ai+b}}{5F_{2ai+b}^2+L_{as}^2}=\sum_{i=0}^{s-1}\frac{1}{L_{a(2i-s)+b}},\ \textit{if } b \textit{ is even and } a \textit{ or } s \textit{ is even,} \tag{9.3}$$

$$L_{as}\sum_{i=0}^{\infty}\frac{F_{2ai+b}}{F_{2ai+b}^2+F_{as}^2}=\sum_{i=0}^{s-1}\frac{1}{F_{a(2i-s)+b}},\ \textit{if } b \textit{ is even and } a \textit{ and } s \textit{ are odd,} \tag{9.4}$$

$$5F_{as}\sum_{i=0}^{\infty}\frac{L_{2ai+b}}{L_{2ai+b}^2+L_{as}^2}=\sum_{i=0}^{s-1}\frac{1}{F_{a(2i-s)+b}},\ \textit{if } b \textit{ is odd and } a \textit{ or } s \textit{ is even,} \tag{9.5}$$

$$L_{as}\sum_{i=0}^{\infty}\frac{L_{2ai+b}}{L_{2ai+b}^2+5F_{as}^2}=\sum_{i=0}^{s-1}\frac{1}{L_{a(2i-s)+b}},\ \textit{if } b \textit{ is odd and } a \textit{ and } s \textit{ are odd.} \tag{9.6}$$

Lemma 9.3. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\frac{\left(t^{as}+t^{-as}\right)\left(t^{2an+b}+t^{-2an-b}\right)}{\left(t^{2an+b}+t^{-2an-b}\right)^2+\left(t^{as}-t^{-as}\right)^2}=\frac{1}{t^{a(2n-s)+b}+t^{-a(2n-s)-b}}+\frac{1}{t^{a(2n+s)+b}+t^{-a(2n+s)-b}}. \tag{9.7}$$

Alternating signs are required for the right side of (9.7) to telescope when summed, and also s must be odd. It then follows from Lemma 9.3 that, for s odd,

$$\sum_{i=0}^{\infty}\frac{(-1)^i\left(t^{as}+t^{-as}\right)\left(t^{2ai+b}+t^{-2ai-b}\right)}{\left(t^{2ai+b}+t^{-2ai-b}\right)^2+\left(t^{as}-t^{-as}\right)^2}=\sum_{i=0}^{s-1}\frac{(-1)^i}{t^{a(2i-s)+b}+t^{-a(2i-s)-b}}. \tag{9.8}$$

Equation (9.8) leads to the following theorem.

Theorem 9.4. *Let $a \geq 1$, $b \geq 0$. Also let $s \geq 1$ be odd. Then*

$$L_{as}\sum_{i=0}^{\infty}\frac{(-1)^iL_{2ai+b}}{L_{2ai+b}^2+5F_{as}^2}=\sum_{i=0}^{s-1}\frac{(-1)^i}{L_{a(2i-s)+b}},\ \textit{if } a \textit{ and } b \textit{ are both even,} \tag{9.9}$$

$$5F_{as}\sum_{i=0}^{\infty}\frac{(-1)^iF_{2ai+b}}{5F_{2ai+b}^2+L_{as}^2}=\sum_{i=0}^{s-1}\frac{(-1)^i}{L_{a(2i-s)+b}},\ \textit{if } a \textit{ and } b \textit{ are both odd,} \tag{9.10}$$

$$5F_{as}\sum_{i=0}^{\infty}\frac{(-1)^iL_{2ai+b}}{L_{2ai+b}^2+L_{as}^2}=\sum_{i=0}^{s-1}\frac{(-1)^i}{F_{a(2i-s)+b}},\ \textit{if } a \textit{ is odd and } b \textit{ is even,} \tag{9.11}$$

$$L_{as}\sum_{i=0}^{\infty}\frac{(-1)^iF_{2ai+b}}{F_{2ai+b}^2+F_{as}^2}=\sum_{i=0}^{s-1}\frac{(-1)^i}{F_{a(2i-s)+b}},\ \textit{if } a \textit{ is even and } b \textit{ is odd.} \tag{9.12}$$

The lemma that follows leads to Theorem 9.6.

Lemma 9.5. *Let $t > 1$ be a real number, and let $n \geq 0$. Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$\frac{t^{2as}-t^{-2as}}{\left(t^{2an+b}-t^{-2an-b}\right)^2+\left(t^{as}+t^{-as}\right)^2}=\frac{t^{-a(2n-s)-b}}{t^{a(2n-s)+b}+t^{-a(2n-s)-b}}-\frac{t^{-a(2n+s)-b}}{t^{a(2n+s)+b}+t^{-a(2n+s)-b}}. \tag{9.13}$$

From Lemma 9.5, we have

$$\sum_{i=0}^{\infty} \frac{t^{2as} - t^{-2as}}{\left(t^{2ai+b} - t^{-2ai-b}\right)^2 + \left(t^{as} + t^{-as}\right)^2} = \sum_{i=0}^{s-1} \frac{t^{-a(2i-s)-b}}{t^{a(2i-s)+b} + t^{-a(2i-s)-b}}. \tag{9.14}$$

Equation (9.14) implies our final theorem, which, as in Theorem 7.3, contains two cases that depend upon the parities of as and b.

Theorem 9.6. *Let $a \geq 1$, $b \geq 0$, and $s \geq 1$. Then*

$$2F_{2as} \sum_{i=0}^{\infty} \frac{1}{5F_{2ai+b}^2 + L_{as}^2} = \frac{s}{\sqrt{5}} - \sum_{i=0}^{s-1} \frac{F_{a(2i-s)+b}}{L_{a(2i-s)+b}}, \text{ if } as \text{ and } b \text{ have the same parity}, \tag{9.15}$$

$$2F_{2as} \sum_{i=0}^{\infty} \frac{1}{F_{2ai+b}^2 + F_{as}^2} = s\sqrt{5} - \sum_{i=0}^{s-1} \frac{L_{a(2i-s)+b}}{F_{a(2i-s)+b}}, \text{ if } as \text{ and } b \text{ have different parities}. \tag{9.16}$$

Equations (9.15) and (9.16) yield, respectively,

$$2F_{2s} \sum_{i=0}^{\infty} \frac{1}{5F_{2i+1}^2 + L_s^2} = \frac{s}{\sqrt{5}} - \sum_{i=0}^{s-1} \frac{F_{2i+1-s}}{L_{2i+1-s}}, \text{ if } s \geq 1 \text{ is odd}, \tag{9.17}$$

$$2F_{2s} \sum_{i=0}^{\infty} \frac{1}{F_{2i}^2 + F_s^2} = s\sqrt{5} - \sum_{i=0}^{s-1} \frac{L_{2i-s}}{F_{2i-s}}, \text{ if } s \geq 1 \text{ is odd}. \tag{9.18}$$

Let $s = 1$. Then (9.17) and (9.18) become, respectively,

$$2\sum_{i=0}^{\infty} \frac{1}{5F_{2i+1}^2 + 1} = \frac{1}{\sqrt{5}}, \tag{9.19}$$

$$2\sum_{i=0}^{\infty} \frac{1}{F_{2i}^2 + 1} = 1 + \sqrt{5}. \tag{9.20}$$

10. Concluding Comments

In this paper we present closed forms for sums, both finite and infinite, where the summand contains Fibonacci/Lucas numbers. We achieve this by first introducing certain algebraic identities. Those algebraic identities that contain only the parameters a and b produce closed forms for finite sums. Upon letting the upper limit $n \to \infty$ in each of these finite sums, we obtain infinite sums, which, to conserve space, we have not written down. The algebraic identities that contain the parameters a, b, and s produce closed forms for infinite sums of a more general nature than those described in the previous sentence.

Our interest in this research was sparked by the response of Almkvist [1] to the ground breaking paper of Backstrom [2]. For more details and references, see [3], where there are also hints regarding how our work might be adapted to more general sequences.

References

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