ANALOGUES OF A FIBONACCI-LUCAS IDENTITY

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ABSTRACT. Sury's 2014 proof of an identity for Fibonacci and Lucas numbers (Identity 236 of Benjamin and Quinn's 2003 book: *Proofs that Count: The Art of Combinatorial Proof*) has excited a lot of comment. We give an alternate, telescoping, proof of this, and associated, identities and generalize them. We also give analogous identities for other sequences that satisfy a three-term recurrence relation.

1. LUCAS' 1876 IDENTITY AND ITS ASSOCIATES

Let F_k and L_k be the kth Fibonacci and kth Lucas numbers, respectively, for $k = 0, 1, 2, \ldots$. Both sequences of numbers satisfy a common recurrence relation, for $n = 0, 1, 2, \ldots$

$$x_{n+2} = x_{n+1} + x_n;$$

with differing initial values: $F_0 = 0, F_1 = 1$ but $L_0 = 2, L_1 = 1$.

Fibonacci numbers satisfy many identities. The classic 1876 identity is due to Lucas himself:

$$F_0 + F_1 + F_2 + \cdots + F_{n-1} = F_{n+1} - 1,$$

which (in view of the relation $F_{k+1} + F_{k-1} = L_k$, for $k \geq 1$) can be written as

$$\sum_{k=0}^{n} (L_k - F_{k+1}) = F_{n+1}, \tag{1.1}$$

where we used $1 = L_0 - F_1$, $F_0 = L_1 - F_2$, $F_1 = L_2 - F_3$, and so on. Next consider Identity 236 of Benjamin and Quinn [4, p. 131]:

$$\sum_{k=0}^{n} 2^k L_k = 2^{n+1} F_{n+1}. \tag{1.2}$$

A recent proof of this identity by Sury [15] excited a lot of comment. Kwong [10] gave an alternate proof, and Marques [11] (see also Martinjak [12]) found an analogous identity that replaces 2 by 3. Marques' identity can be written (in the form presented by Martinjak):

$$\sum_{k=0}^{n} 3^k (L_k + F_{k+1}) = 3^{n+1} F_{n+1}. \tag{1.3}$$

On the lines of (1.2), Martinjak gave an identity with alternating signs:

$$\sum_{k=0}^{n} \frac{(-1)^k}{2^k} L_{k+1} = \frac{(-1)^n}{2^n} F_{n+1}. \tag{1.4}$$

Finally, the other classic identity (we are unaware of its provenance) which consists of alternating sums of Fibonacci numbers, see Koshy [9, Identity 20, p. 88]:

$$F_2 - F_3 + F_4 + \cdots + (-1)^{n+2}F_{n+2} = (-1)^n F_{n+1},$$

can be written as

$$\sum_{k=0}^{n}(-1)^k \left(L_{k+1} - F_k\right) = (-1)^n F_{n+1}. \tag{1.5}$$

Now many of the classic Fibonacci identities follow from telescoping, including Lucas' classic identity (1.1). And all telescoping formulas can be derived from what we [5] called Euler's Telescoping Lemma, see (2.1) below. If we apply this approach to prove the above identities, we are quickly led to the following generalization, for t an indeterminate:

$$\sum_{k=0}^{n} t^k \left(L_k + (t-2)F_{k+1}\right) = t^{n+1} F_{n+1}. \tag{1.6}$$

Observe that when $t = 1, 2$, and 3, then (1.6) reduces to (1.1), (1.2), and (1.3), respectively. What is remarkable is that (1.6) also contains as special cases the identities (1.4) and (1.5). We can see this by replacing t by $-1/t$, and setting $t = -1$, respectively. But in this case one has to massage the sums a bit using the relations between Fibonacci and Lucas numbers.

Alternatively, one can directly prove for $t \neq 0$:

$$\sum_{k=0}^{n} \frac{(-1)^k}{t^k} \left(L_{k+1} + (t-2)F_k\right) = \frac{(-1)^n}{t^n} F_{n+1}. \tag{1.7}$$

When $t = 1$ and 2, then (1.7) reduces to (1.4) and (1.5), respectively. It is a nice exercise to show that (1.6) and (1.7) are equivalent.

The purpose of this paper is to generalize the above identities to sequences which satisfy a recurrence relation of the form

$$x_{n+2} = a_n x_{n+1} + b_n x_n,$$

where a_n and b_n are sequences of indeterminates, which can be specialized to complex numbers or polynomials when required. As examples, in addition to (1.6) and (1.7), we will note many analogous identities for other sequences that are defined by this recurrence relation. See [5] for more examples of this approach.

2. Euler's Telescoping Lemma and its Application

Euler's Telescoping Lemma can be written as [5, equation (2.2)]:

$$\sum_{k=1}^{n} w_k \frac{u_1 u_2 \cdots u_{k-1}}{v_1 v_2 \cdots v_k} = \frac{u_1 u_2 \cdots u_n}{v_1 v_2 \cdots v_n} - 1, \tag{2.1}$$

where $w_k = u_k - v_k$. Here the product $u_1 \cdots u_{k-1}$ is considered to be equal to 1 when $k = 1$. The u_k and v_k are sequences of indeterminates. Its proof is immediate. Replace w_k by $u_k - v_k$ in the sum on the left-hand side, and (2.1) follows by telescoping. We showed in [5] that an equivalent form of this identity characterizes telescoping sums. Thus all telescoping sums can be obtained as suitable special cases.

For example, to obtain (1.6) set $u_k = tF_{k+1}$, $v_k = F_k$. Then

$$
\begin{aligned}
w_k &= u_k - v_k \\
&= tF_{k+1} - F_k \\
&= (F_{k+1} - F_k) + (t-1)F_{k+1} \\
&= F_{k-1} + (t-1)F_{k+1} \\
&= (F_{k-1} + F_{k+1}) + (t-2)F_{k+1} \\
&= L_k + (t-2)F_{k+1}.
\end{aligned}
$$

Now substitute for u_k, v_k, and w_k in (2.1) to obtain (1.6).

Similarly, to obtain (1.7), set $u_k = F_{k+1}$, $v_k = -tF_k$ in (2.1) and observe that $w_k = L_{k+1} + (t-2)F_k$.

For sequences satisfying a more general three-term recurrence relation, we have the following generalization.

Theorem 2.1. *Let a_n and b_n be sequences, with $a_n \neq 0$, $b_n \neq 0$, for all n. Let t be an indeterminate. Consider a sequence x_n that satisfies (for $n \geq 0$):*

$$
x_{n+2} = a_n x_{n+1} + b_n x_n.
$$

Then the following identities hold for $n = 0, 1, 2, \ldots$, provided the denominators are not 0.

$$
\sum_{k=1}^{n} \frac{t^{k-1}}{a_0 a_1 \cdots a_{k-1}} \frac{b_{k-1}x_{k-1} + (t-1)x_{k+1}}{x_1} = \frac{t^n}{a_0 a_1 \cdots a_{n-1}} \frac{x_{n+1}}{x_1} - 1. \tag{2.2}
$$

$$
\sum_{k=1}^{n} \frac{(-1)^k}{t^k} \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} \frac{x_{k+2} + (t-1)b_k x_k}{x_1} = \frac{(-1)^n}{t^n} \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \frac{x_{n+1}}{x_1} - 1. \tag{2.3}
$$

Remarks. The special case $t = 1$ of identity (2.3) has appeared previously in [5, equation (10.11)]. When $a_k = 1 = b_k$, then (2.2) reduces to (1.6), and (2.3) reduces to (1.7).

Proof. We derive (2.2) from Euler's Telescoping Lemma (2.1). Set $u_k = tx_{k+1}$, $v_k = a_{k-1}x_k$. Note that $w_k = (b_{k-1}x_{k-1} + (t-1)x_{k+1})$. Substituting in (2.1), we immediately obtain (2.2).

Next, set $u_k = a_k x_{k+1}$ and $v_k = -tb_k x_k$, so that $w_k = (x_{k+2} + (t-1)b_k x_k)$. Substituting in (2.1), we obtain (2.3). □

Next, we note some examples of sequences that satisfy a three term recurrence relation.

3. Examples of Analogous Identities

First consider the kth Pell number P_k and the kth Pell-Lucas number Q_k. Both these sequences satisfy the recurrence relation, for $n = 0, 1, 2, \ldots$

$$
x_{n+2} = 2x_{n+1} + x_n,
$$

and have initial values: $P_0 = 0$, $P_1 = 1$; and, $Q_0 = 2 = Q_1$. It is easy to check that P_k and Q_k satisfy the relation

$$
P_{k+1} + P_{k-1} = Q_k, \text{ for } k \geq 1.
$$

Substituting $a_k = 2$, $b_k = 1$, and $x_k = P_k$ in (2.2), multiplying both sides by t, and using the above relation between the Pell and Pell-Lucas numbers, we obtain a Pell analogue of (1.6):

$$
\sum_{k=0}^{n} \frac{t^k}{2^k}(Q_k + (t-2)P_{k+1}) = \frac{t^{n+1}}{2^n}P_{n+1}.
$$

When t is replaced by $2t$, we have

$$\sum_{k=0}^{n} t^k \left(Q_k + 2(t-1)P_{k+1}\right) = 2t^{n+1}P_{n+1}. \tag{3.1}$$

Similarly, equation (2.3) yields, after replacing t by $2t$, for $t \neq 0$:

$$\sum_{k=0}^{n} \frac{(-1)^k}{2t^k} \left(Q_{k+1} + 2(t-1)P_k\right) = \frac{(-1)^n}{t^n}P_{n+1}. \tag{3.2}$$

Compare the $t = 1$ case of (3.1)

$$\sum_{k=0}^{n} Q_k = 2P_{n+1}. \tag{3.3}$$

with (1.1) and (1.2). The identity

$$\sum_{k=0}^{n} (-1)^k Q_{k+1} = (-1)^n 2P_{n+1}, \tag{3.4}$$

obtained by setting $t = 1$ in (3.2) is similarly very appealing.

Next, we obtain identities for the Lucas numbers. Recall that Lucas numbers also satisfy the same recursion as the Fibonacci numbers, and have initial values $L_0 = 2$ and $L_1 = 1$. In this case, (2.2) yields:

$$1 + \sum_{k=0}^{n} t^k \left(L_{k-1} + (t-1)L_{k+1}\right) = t^{n+1}L_{n+1}, \tag{3.5}$$

where we have taken $L_{-1} = 0$ in order to write the sum from 0 to n. Similarly, equation (2.3) yields, for $t \neq 0$:

$$1 + \sum_{k=1}^{n} \frac{(-1)^k}{t^k} \left(L_{k+2} + (t-1)L_k\right) = \frac{(-1)^n}{t^n}L_{n+1}. \tag{3.6}$$

The Derangement Numbers d_n, counting the number of derangements—that is, the number of permutations on n letters with no fixed points—are usually not considered to be analogous to the Fibonacci numbers. However, they do satisfy a three term recurrence relation. Since the formulas (2.2) and (2.3) involve division by x_1, and $d_1 = 0$, we have to make an adjustment. We consider the shifted derangement numbers D_n which satisfy $D_n = d_{n+1}$. They are defined by: $D_0 = 0$, $D_1 = 1$ and for $n \geq 0$

$$D_{n+2} = (n+2)D_{n+1} + (n+2)D_n.$$

In this case $a_k = k + 2 = b_k$. Equation (2.2) gives us

$$1 + \sum_{k=1}^{n} \frac{t^{k-1}}{(k+1)!} \left((k+1)D_{k-1} + (t-1)D_{k+1}\right) = \frac{t^n}{(n+1)!}D_{n+1}. \tag{3.7}$$

Similarly, equation (2.3) yields, for $t \neq 0$:

$$\sum_{k=0}^{n} \frac{(-1)^k}{t^k} \left(\frac{D_{k+2} + (t-1)(k+2)D_k}{k+2}\right) = \frac{(-1)^n}{t^n}D_{n+1}. \tag{3.8}$$

Identities for the derangement numbers are obtained by replacing D_n by d_{n+1}.

Next, we consider the q-Fibonacci numbers considered first by Schur, and later studied by Andrews [1, 2, 3], Carlitz [6, 7], and Smith [14]. We use the notation of Garrett [8] who has

studied these sequences combinatorially. The q-Fibonacci numbers are defined by $F_0^{(a)}(q) = 0$, $F_1^{(a)}(q) = 1$, and

$$F_{n+2}^{(a)}(q) = F_{n+1}^{(a)}(q) + q^{a+n} F_n^{(a)}(q).$$

In this case $a_k = 1$, $b_k = q^{a+k}$ and (2.2) yields

$$1 + \sum_{k=1}^{n} t^{k-1} \left(q^{a+k-1} F_{k-1}^{(a)}(q) + (t-1) F_{k+1}^{(a)}(q) \right) = t^n F_{n+1}^{(a)}(q). \tag{3.9}$$

Similarly, equation (2.3) yields, for $t \neq 0$:

$$\sum_{k=0}^{n} \frac{(-1)^k}{t^k} q^{-ak} q^{-\binom{k+1}{2}} \left(F_{k+2}^{(a)}(q) + (t-1) q^{a+k} F_k^{(a)}(q) \right)$$

$$= \frac{(-1)^n}{t^n} q^{-na} q^{-\binom{n+1}{2}} F_{n+1}^{(a)}(q). \tag{3.10}$$

The above identities may be regarded as q-analogs of (1.6) and (1.7). As a final set of examples, we give another set of q-analogs of these two identities.

We need the notation for q-rising factorials. The q-rising factorial (for q a complex number) is defined as the product:

$$(a; q)_m := \begin{cases} 1 & \text{if } m = 0, \\ (1-a)(1-aq) \cdots (1 - aq^{m-1}) & \text{if } m \geq 1. \end{cases}$$

In Euler's Telescoping Lemma (2.1), consider $u_k = (1 - tq^{a+k-1}) F_{k+1}^{(a)}(q)$ and $v_k = F_k^{(a)}(q)$. Then

$$w_k = q^{a+k-1} \left(F_{k-1}^{(a)}(q) - t F_{k+1}^{(a)}(q) \right),$$

and we obtain:

$$1 + \sum_{k=1}^{n} q^{a+k-1}(tq^a; q)_{k-1} \left(F_{k-1}^{(a)}(q) - t F_{k+1}^{(a)}(q) \right) = (tq^a; q)_n F_{n+1}^{(a)}(q). \tag{3.11}$$

Next in (2.1), set $u_k = F_{k+1}^{(a)}(q)$ and $v_k = (-1) q^{k+a}(1 - tq^{-(a+k)}) F_k^{(a)}(q)$. Then

$$w_k = F_{k+2}^{(a)}(q) - t F_k^{(a)}(q),$$

and we obtain, after some simplification:

$$\sum_{k=0}^{n} \frac{1}{t^k} \frac{F_{k+2}^{(a)}(q) - t F_k^{(a)}(q)}{(t^{-1} q^{a+1}; q)_k} = \frac{1}{t^n} \frac{F_{n+1}^{(a)}(q)}{(t^{-1} q^{a+1}; q)_n} \tag{3.12}$$

By now, the reader may well think that we have strayed quite far from the classic identities (1.1) and (1.5). However, given the uniform way they have been derived, one can see that all the identities presented here are related to each other. We emphasize that the above are only a small set of possible analogues. There are many sequences of numbers, and of polynomials, that satisfy such a three term recurrence. They all have analogues of Lucas' identity and its associates.

4. What's the Point?

Any identity of the type

$$\sum_{k=0}^{n} t_k = T_n$$

is a telescoping sum, since one can write $T_k - T_{k-1} = t_k$ and then sum both sides. But how do you make it telescope? For many identities, we can use the WZ method (see Petkovšek, Wilf and Zeilberger [13]) and let the computer find the telescoping. For sequences defined by a three term recurrence relation, this may not always be possible, or practical. But you can use Euler's Telescoping Lemma to find the telescoping, without using a computer! This is the point of [5], which gives many examples of this approach, including several involving analogues of the Fibonacci numbers.

So if you have found a Fibonacci type identity, you can use Euler's Telescoping Lemma to find analogous identities for other interesting sequences.

Note Added to proof. After submitting this paper, we found that T. Edgar (The Fibonacci Quarterly **54.1** (2016), 79) has independently given identity (1.6) for the case where t is a natural number bigger than 1.

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