DETERMINANTS CONTAINING RISING POWERS OF FIBONACCI NUMBERS

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Abstract. A matrix containing rising powers of Fibonacci numbers is investigated. The LU-decomposition is guessed and proved; this leads to a formula for the determinant. Similar results are also obtained for a matrix of Lucas numbers.

1. Introduction

Carlitz [1], motivated by earlier writings computed the determinant

$$\begin{vmatrix} F_n^r & F_{n+1}^r & F_{n+2}^r & \cdots & \\ F_{n+1}^r & F_{n+2}^r & F_{n+3}^r & \cdots & \\ F_{n+2}^r & F_{n+3}^r & F_{n+4}^r & \cdots & \\ \cdots & \cdots & \cdots & \ddots & \\ & & & & F_{n+2r}^r \end{vmatrix},$$

with the result

$$(-1)^{\binom{r+1}{2}(n+1)} \prod_{j=0}^{r} \binom{r}{j} \cdot (F_1^r F_2^{r-1} \dots F_r)^2;$$

F_i are Fibonacci numbers as usual, and r and n are non-negative integers.

In the present note we consider the rising powers analogue

$$M = \begin{pmatrix} F_n^{\langle r\rangle} & F_{n+1}^{\langle r\rangle} & F_{n+2}^{\langle r\rangle} & \cdots & \\ F_{n+1}^{\langle r\rangle} & F_{n+2}^{\langle r\rangle} & F_{n+3}^{\langle r\rangle} & \cdots & \\ F_{n+2}^{\langle r\rangle} & F_{n+3}^{\langle r\rangle} & F_{n+4}^{\langle r\rangle} & \cdots & \\ \cdots & \cdots & \cdots & \ddots & \\ & & & & F_{n+2r}^{\langle r\rangle} \end{pmatrix}.$$

This is an $(r+1) \times (r+1)$ matrix, and we assume that the indices i and j of $M_{i,j}$ run from $0, \dots, r$. The rising products are defined as follows:

$$F_m^{\langle r\rangle} := F_m F_{m+1} \cdots F_{m+r-1}.$$

Although this definition looks more complicated than the one used by Carlitz, it is actually nicer, since we are able to compute (first guessing, then proving) the LU-decomposition of $M = LU$, from which the determinant is an easy corollary, via $\det(M) = U_{0,0}U_{1,1} \dots U_{r,r}$.

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2. The LU-Decomposition of M

We start from the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad q = \frac{\beta}{\alpha} = -\frac{1}{\alpha^2},$$

so that $\alpha = \mathrm{i}q^{-1/2}$. We write further

$$F_{n+j} = y\alpha^{j-1}\frac{1-xq^j}{1-q},$$

with

$$y = \alpha^n \quad \text{and} \quad x = q^n.$$

Thus,

$$M_{i,j} = F_{n+i+j}^{\langle r\rangle} = \frac{y^r}{(1-q)^r}\alpha^{(i+j-1)r+\binom{r}{2}}(xq^{i+j};q)_r.$$

Here, we use standard q-notation: $(x;q)_m := (1-x)(1-xq)\cdots(1-xq^{m-1})$.

This is the form that we use to guess (and then prove) the LU-decomposition. It holds for general variables x, y, q, α. However, for our application, we will then specialize. For these specializations, we need the notation of a Fibonacci-factorial:

$$n!_F := F_1F_2\ldots F_n.$$

Theorem 2.1. *For $0 \le i \le j \le r$,*

$$U_{i,j} = \frac{x^iy^r}{(1-q)^r}\alpha^{r(i+j)+\frac{r(r-3)}{2}}q^{\frac{3(i-1)i}{2}}(-1)^i\frac{(x;q)_{j+r}(x;q)_{i-1}(q;q)_j(q;q)_r}{(x;q)_{i+j}(x;q)_{2i-1}(q;q)_{r-i}(q;q)_{j-i}}.$$

For $0 \le j \le i \le r$,

$$L_{i,j} = \frac{(x;q)_{i+r}(q;q)_i(x;q)_{2j}}{(x;q)_{j+r}(x;q)_{i+j}(q;q)_j(q;q)_{i-j}}\alpha^{r(i-j)}.$$

Corollary 1. *The specialized versions (Fibonacci numbers) are as follows:*

$$U_{i,j} = \frac{(n+j+r-1)!_F\,(n+i-2)!_F\,j!_F\,r!_F}{(n+i+j-1)!_F\,(n+2i-2)!_F\,(r-i)!_F\,(j-i)!_F}(-1)^{\frac{i(i+1)}{2}+ni},$$

$$L_{i,j} = \frac{(n+i+r-1)!_F\,(n+2j-1)!_F\,i!_F}{(n+j+r-1)!_F\,(n+i+j-1)!_F\,j!_F\,(i-j)!_F}.$$

Theorem 2.2. *The determinant of the matrix M is given by*

$$\begin{aligned}\det(M) &= \prod_{i=0}^{r} U_{i,i} = (-1)^{\binom{r+2}{3}+n\binom{r+1}{2}}(r!_F)^{r+1}\prod_{i=0}^{r}\frac{(n+i+r-1)!_F\,(n+i-2)!_F}{(n+2i-1)!_F\,(n+2i-2)!_F}\\ &= (-1)^{\binom{r+2}{3}+n\binom{r+1}{2}}(r!_F)^{r+1}.\end{aligned}$$

Although it is not necessary for our determinant calculation, we briefly mention two additional results (first general, then specialized).

Theorem 2.3.

$$
\begin{aligned}
U_{i,j}^{-1} &= \frac{(q;q)_{2j}(q;q)_{r-j}(x;q)_{i+j-1}}{(q;q)_i(q;q)_r(q;q)_{j-i}(x;q)_{j-1}(x;q)_{i+r}} \\
&\quad\times q^{-j(j-1)-ij+\frac{(i+1)i}{2}}(-1)^i(1-q)^r\alpha^{-r(i+j)-\frac{r(r-3)}{2}}x^{-j}y^{-r}, \\
L_{i,j}^{-1} &= \frac{(x;q)_{i+r}(x;q)_{i+j-1}(q;q)_i}{(x;q)_{j+r}(x;q)_{2i-1}(q;q)_j(q;q)_{i-j}}q^{\frac{i(i-1)}{2}-ij+\frac{(j+1)j}{2}}\alpha^{r(i-j)}(-1)^{i-j}, \\
U_{i,j}^{-1} &= \frac{(n+2j-1)!_F\,(n+i+j-2)!_F\,(r-j)!_F}{(n+j-2)!_F\,(n+i+r-1)!_F\,r!_F\,(j-i)!_F\,i!_F}(-1)^{ij+\frac{i(i-1)}{2}+rj}, \\
L_{i,j}^{-1} &= \frac{(n+i+r-1)!_F\,(n+i+j-2)!_F\,i!_F}{(n+j+r-1)!_F\,(n+2i-2)!_F\,j!_F\,(i-j)!_F}(-1)^{\frac{i(i-1)}{2}+ij+\frac{j(j+1)}{2}}.
\end{aligned}
$$

3. Sketch of Proof

To check that $M = L\cdot U$, we consider an arbitrary element $(L\cdot U)_{i,k}$.
We must simplify the following sum:

$$
\begin{aligned}
\sum_j L_{i,j}U_{j,k} = \sum_j &\frac{(x;q)_{i+r}(q;q)_i(x;q)_{2j}}{(x;q)_{j+r}(x;q)_{i+j}(q;q)_j(q;q)_{i-j}}\alpha^{r(i-j)} \\
&\times\frac{x^jy^r}{(1-q)^r}\alpha^{r(j+k)+\frac{r(r-3)}{2}}q^{\frac{3(j-1)j}{2}}(-1)^j \\
&\times\frac{(x;q)_{k+r}(x;q)_{j-1}(q;q)_k(q;q)_r}{(x;q)_{j+k}(x;q)_{2j-1}(q;q)_{r-j}(q;q)_{k-j}}.
\end{aligned}
$$

Apart from constant factors, we are left to compute

$$
\begin{aligned}
&\sum_{j=0}^{\min\{i,k\}} x^j(-1)^jq^{\frac{3(j-1)j}{2}} \\
&\quad\times\frac{(x;q)_{2j}(x;q)_{j-1}}{(x;q)_{j+r}(x;q)_{i+j}(x;q)_{j+k}(x;q)_{2j-1}(q;q)_j(q;q)_{i-j}(q;q)_{r-j}(q;q)_{k-j}}.
\end{aligned}
$$

Zeilberger's algorithm [2] (the q-version of it) readily evaluates this as

$$
\frac{(x;q)_{i+k+r}}{(x;q)_{i+r}(x;q)_{k+r}(x;q)_{i+k}(q;q)_r(q;q)_i(q;q)_k}.
$$

Putting this together with the constant factors proves that $LU = M$.

There now exist many implementations of this important algorithm, notably by Zeilberger himself, based on the computer algebra system Maple. It is freely available from Zeilberger's homepage. This was the program of our choice.

4. The Lucas Matrix

We briefly discuss the case of the matrix $\mathcal{M}$, where each F_i is replaced by the Lucas number L_i. We also need the notation $m!_L := L_1L_2\dots L_m$.

We write $L_m = \alpha^m+\beta^m = \alpha^m(1+q^m)$ and $L_{n+j} = y\alpha^j(1+xq^j)$, with $y=\alpha^n$ and $x=q^n$, when it comes to specializations.

Theorem 4.1. *The LU-decomposition $\mathcal{M} = \mathcal{L}\mathcal{U}$ is given by:*

$$\begin{aligned}
\mathcal{U}_{i,j} &= \frac{(-x;q)_{j+r}(-x;q)_{i-1}(q;q)_j(q;q)_r}{(q;q)_{j-i}(-x;q)_{i+j}(q;q)_{r-i}(-x;q)_{2i-1}} x^i y^r q^{\frac{3i(i-1)}{2}} \alpha^{r(i+j)+\frac{r(r-1)}{2}},\\
\mathcal{L}_{i,j} &= \frac{(-x;q)_{i+r}(-x;q)_{2j}(q;q)_i}{(-x;q)_{j+r}(-x;q)_{i+j}(q;q)_j(q;q)_{i-j}} \alpha^{r(i-j)},\\
\mathcal{U}^{-1}_{i,j} &= \frac{(-x;q)_{2j}(-x;q)_{i+j-1}(q;q)_{r-j}}{(-x;q)_{j-1}(-x;q)_{i+r}(q;q)_r(q;q)_i(q;q)_{j-i}}\\
&\quad \times x^{-j} y^{-r} q^{-j(j-1)-ij+\frac{i(i+1)}{2}} \alpha^{-r(i+j)+\frac{r(r-9)}{2}} (-1)^{i-j},\\
\mathcal{L}^{-1}_{i,j} &= \frac{(-x;q)_{i+r}(-x;q)_{i+j-1}(q;q)_i}{(-x;q)_{j+r}(-x;q)_{2i-1}(q;q)_j(q;q)_{i-j}} q^{\frac{i(i-1)}{2}-ij+\frac{j(j+1)}{2}} \alpha^{r(i-j)}(-1)^{i-j}.
\end{aligned}$$

Theorem 4.2. *The specialized (Fibonacci/Lucas) forms are:*

$$\begin{aligned}
\mathcal{U}_{i,j} &= \frac{(n+j+r-1)!_L\,(n+i-2)!_L\,j!_F\,r!_F}{(n+i+j-1)!_L\,(n+2i-2)!_L\,(j-i)!_F\,(r-i)!_F} 5^i (-1)^{\frac{i(i-1)}{2}+ni},\\
\mathcal{L}_{i,j} &= \frac{(n+i+r-1)!_L\,(n+2j-1)!_L\,i!_F}{(n+j+r-1)!_L\,(n+i+j-1)!_L\,j!_F\,(i-j)!_F},\\
\mathcal{U}^{-1}_{i,j} &= \frac{(n+2j-1)!_L\,(n+i+j-2)!_L\,(r-j)!_F}{(n+j-2)!_L\,(n+i+r-1)!_L\,r!_F\,(j-i)!_F\,i!_F} 5^{-j}(-1)^{ij+\frac{i(i-1)}{2}+(n+1)j},\\
\mathcal{L}^{-1}_{i,j} &= \frac{(n+i+r-1)!_L\,(n+i+j-2)!_L\,i!_F}{(n+j+r-1)!_L\,(n+2i-2)!_L\,j!_F\,(i-j)!_F} (-1)^{\frac{i(i+1)}{2}+ji+\frac{j(j-1)}{2}}.
\end{aligned}$$

Theorem 4.3. *The determinant of the matrix $\mathcal{M}$ is given by*

$$\begin{aligned}
\det(\mathcal{M}) &= \prod_{i=0}^{r} U_{i,i} = \prod_{i=0}^{r} \frac{(n+i+r-1)!_L\,(n+i-2)!_L\,i!_F\,r!_F}{(n+2i-1)!_L\,(n+2i-2)!_L\,(r-i)!_F} 5^i (-1)^{\frac{i(i-1)}{2}+ni}\\
&= (r!_F)^{r+1} 5^{\binom{r+1}{2}} (-1)^{\binom{r+1}{3}+n\binom{r+1}{2}}. \quad \square
\end{aligned}$$

References

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[2] M. Petkovšek, H. Wilf, and D. Zeilberger, $A = B$, A. K. Peters, Ltd., Wellesley, MA, 1996.

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