A COMBINATORIAL IDENTITY RELATED TO CROSS POLYTOPE NUMBERS

STEVEN EDWARDS AND WILLIAM GRIFFITHS

Abstract. We give a combinatorial argument to prove a new identity for binomial coefficients. In the course of the proof, we discover two new closed forms for the figurate numbers for the n-dimensional cross polytopes.

1. Introduction

The main result of this paper is a new identity for the binomial coefficients:

$$\binom{n}{k} = \binom{2n+k+1}{k} + \sum_{j=1}^{k} (-1)^j \frac{n+k+1}{j} \binom{n+k-j}{j-1} \binom{2n+k+1-2j}{k-j}. \tag{1}$$

We prove this identity combinatorially, beginning by using a method similar to the principle of inclusion and exclusion, but veering away from this to a more complicated argument. To complete this argument, we will use the figurate numbers for the n-dimensional cross polytopes. Along the way, we discover two new closed forms for the polytope numbers.

We begin with our discovery of (1). In [4], the problem posed was to find a closed form for

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} F_{2n-3k}.$$

Calculations with small values of n lead one to suspect that the closed form is F_n. If the *rising diagonal* sum from Pascal's Triangle is used,

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i},$$

then

$$\begin{aligned}
&\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} F_{2n-3k} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \sum_{i=0}^{\lfloor \frac{2n-3k-1}{2} \rfloor} \binom{2n-3k-i-1}{i} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{\lfloor \frac{2n-3k-1}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \binom{2n-3k-i-1}{i}.
\end{aligned}$$

Re-indexing $j = i + k$, and noting that for larger values of j and k many binomial coefficients are zero, gives

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j \frac{n}{n-j}\binom{n-j}{j}\binom{2n-k-1-2j}{k-j}.$$

In the *rising diagonal* formula, the upper limit of the sum can be increased, since doing so gives terms which equal zero. This implies that the original expression then equals F_n if

$$\sum_{j=0}^{k}(-1)^j \frac{n}{n-j}\binom{n-j}{j}\binom{2n-k-1-2j}{k-j} = \binom{n-k-1}{k}.$$

Replacing $n-k-1$ by n gives the equivalent equation

$$\binom{n}{k} = \sum_{j=0}^{k}(-1)^j \frac{n+k+1}{n+k+1-j}\binom{n+k+1-j}{j}\binom{2n+k+1-2j}{k-j}.$$

Pulling off the first term and using $\frac{m}{j}\binom{m-1}{j-1} = \binom{m}{j}$ produces the identity we wish to prove:

$$\binom{n}{k} = \binom{2n+k+1}{k} + \sum_{j=1}^{k}(-1)^j \frac{n+k+1}{j}\binom{n+k-j}{j-1}\binom{2n+k+1-2j}{k-j}.$$

We will show that both sides of this identity count the same thing, namely the number of k-element subsets of an n-element set. The right hand side does this in a very strange fashion.

2. Counting by Adding Bad Elements

We will consider our n-element set to be the first positive n integers, and we will call these elements *good*. We then expand this set by adding $n + k + 1$ *bad* elements, the integers from $n + 1$ to $2n + k + 1$. The first term on the right of (1) is then the number of k-element subsets of our expanded set, including all *good combinations* (those with only good elements), plus *bad combinations* (those which contain at least one bad element).

The $j = 1$ term in (1) is

$$(n+k+1)\binom{n+k-1}{0}\binom{2n+k-1}{k-1}. \tag{2}$$

We consider (2) to count certain bad combinations, as follows. The first factor counts the number of ways of choosing one bad element from the $n + k + 1$ bad elements. The next two factors choose from the bad elements and from the expanded set, respectively, but in both cases the sets are reduced. The reductions depend on which element was chosen by the first factor: if $n + 1$ was chosen first, then $n + 2$ is removed; otherwise, $n + 1$ is removed. The second factor counts the number of ways of choosing 0 elements from the reduced bad ones, and the third factor counts the number of ways of choosing $k - 1$ elements from the reduced expanded set. The expression (2) does not count any good combinations. It counts some bad ones once or multiple times, but it does not count combinations with both $n + 1$ and $n + 2$ at all. In total, the three factors select a total of k elements.

Consider now the summand for general j:

$$\frac{n+k+1}{j}\binom{n+k-j}{j-1}\binom{2n+k+1-2j}{k-j}. \tag{3}$$

Our scheme is similar to that for the $j = 1$ term, but with a twist. For the first factor, the smallest bad element allowed is $n + j$, but a combination that has $n + j$ as its smallest bad element is counted j times. This overcounting is corrected by the j in the denominator of the first factor. If $n + j$ is counted by the first factor, then $n + j + 1$ is removed from further choices. If an element larger than $n + j$ is chosen first, then $n + j$ is removed. Counting in this fashion, the jth term, for $1 \leq j \leq k$, does not count a combination with $n + j$ and $n + j + 1$ as the smallest bad elements.

Lemma 2.1. *A combination whose bad elements are all larger than $n + k$ is counted net zero times by the right-hand side of (1).*

Proof. Let such a combination contain m bad elements. Then for the jth term there are m choices for the first factor and $\binom{m-1}{j-1}$ choices for the second factor, so the jth term counts it $\frac{m}{j}\binom{m-1}{j-1} = \binom{m}{j}$ times. This gives a total count of $\sum_{j=0}^{m}(-1)^j\binom{m}{j}$, but it is well-known that this sum is zero. □

Lemma 2.2. *A combination which has $n + i$ as the smallest bad element but which does not contain $n + i + 1$ is counted net zero times by the right-hand side of (1), for $1 \leq i \leq k$.*

Proof. Let such a combination contain $l + 1$ total bad elements, the smallest being $n + i$. The term $\binom{2n+k+1}{1}$ counts this combination one time. The $j = 1$ term subtracts it $l + 1 = \binom{l+1}{1}$ times, once per bad element. The $j = 2$ term adds it back $\frac{l+1}{2}\binom{l}{1} = \binom{l+1}{2}$ times, etc., until the ith term. A combination with $n + i$ will be counted by the ith term only if it is chosen by the first factor, $n + k + 1$. Such a combination is counted i times, but then also divided by i. To complete this combination we can choose from the other l bad elements $\binom{l}{i-1}$ ways with the second factor. Further terms (with $j \geq i$) in the sum will not count this combination because the smallest element in a combination counted by summand j is $n + j$. Thus such a combination will be counted a total of

$$\sum_{m=0}^{i-1}\left[(-1)^m\binom{l+1}{m}\right] + (-1)^i\binom{l}{i-1}$$

times, but as is well-known (see e.g. [1]),

$$\sum_{m=0}^{i-1}\left[(-1)^m\binom{l+1}{m}\right] + (-1)^i\binom{l}{i-1} = 0.$$

□

A combination that has $n + i$ and $n + i + 1$ as the smallest bad elements is an exception to the two lemmas above, for $1 \leq i \leq k$. Each such combination is not counted net zero times. Rather, some combinations of this type contribute positively to the sum (1), and some negatively. Our next theorem will show that the sum of the contributions for all such combinations is zero. Such a combination has at least two bad elements. We proceed by grouping the combinations by their number of bad elements, which can be between 2 and k inclusive.

Lemma 2.3. *The net count in (1) of combinations in which the smallest bad elements are consecutive, with the first bad element between $n + 1$ and $n + k$, is*

$$\sum_{l=0}^{k-2}\binom{n}{k-l-2}\sum_{m=0}^{k-1}(-1)^m\binom{l+1}{m}\binom{n+k-1-m}{l}. \tag{4}$$

Proof. Since each such combination has at least two bad elements, it is convenient to denote the total number of bad elements in such a combination by $l+2$. The number of times each such combination is counted is determined by the value of the smallest bad element. If both $n+1$ and $n+2$ are in such a combination, then that combination is counted once in the term $\binom{2n+k+1}{k}$ and then no more. The number of these combinations with l bad elements besides $n+1$ and $n+2$ is $\binom{n}{k-l-2}\binom{n+k-1}{l}$. Next, a combination with smallest bad elements $n+2$ and $n+3$ is counted likewise once, subtracted once for each bad element, and then no more, i.e. $1-(l+2)$ times. There are $\binom{n}{k-l-2}\binom{n+k-2}{l}$ such combinations.

More generally, the number of combinations that contain $n+m+1$ and $n+m+2$ as the smallest bad elements and $l+2$ bad elements is $\binom{n}{k-l-2}\binom{n+k-1-m}{l}$. Each of these combinations is counted

$$1-(l+2)+\binom{l+2}{2}-\binom{l+2}{3}+\cdots+(-1)^m\binom{l+2}{m}$$

times, which expression, as before, equals $(-1)^m\binom{l+1}{m}$. Hence our count is

$$\begin{aligned}&\sum_{m=0}^{k-1}\sum_{l=0}^{k-2}\binom{n}{k-l-2}\binom{n+k-1-m}{l}(-1)^m\binom{l+1}{m}\\&=\sum_{l=0}^{k-2}\binom{n}{k-l-2}\sum_{m=0}^{k-1}(-1)^m\binom{l+1}{m}\binom{n+k-1-m}{l}.\end{aligned}$$

□

We will show that (4) is zero by showing that the inner summation is zero.

To simplify the notation, we replace $n+k-1$ by n, and $l+1$ by l, noting that terms are non-zero only when $m \le l$. This gives the inner sum as

$$\sum_{m=0}^{l}(-1)^m\binom{l}{m}\binom{n-m}{l-1}. \tag{5}$$

We will show that the sum of the even (positive) terms is equal in absolute value to the sum of the odd (negative) terms.

3. Cross Polytope Numbers

To conclude our argument, we relate our inner sum terms to a seemingly unrelated sequence; that of the figurate numbers for n-dimensional cross polytopes. The polytope numbers are constructed inductively by taking a cross polytope, which is a regular, convex geometric figure, and extending its edges to form a larger cross polytope of similar shape, and counting its vertices [3]. These numbers, which we denote by $T(n,k)$, are defined by $T(n,1)=1$, $T(1,k)=k$, and

$$T(n,k)=T(n,k-1)+T(n-1,k-1)+T(n-1,k)$$

for $n,k\ge 2$ [5]. Let $E(n,l)$ be the sum of the even-indexed terms in (5), and let $O(n,l)$ be the sum of the absolute value of the odd terms. Using the cross polytope numbers, we will show that $E(n,l)=O(n,l)$, from which it follows that the sum in (5) is zero.

Theorem 3.1. *For $n\ge 1$ and $1\le l\le n$,*

$$E(n,l)=O(n,l)=T(n-l+1,l). \tag{6}$$

Proof. For the even and odd terms in (5) we have

$$E(n,l) = \sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} \binom{l}{2m}\binom{n-2m}{l-1} \quad \text{and} \quad O(n,l) = \sum_{m=0}^{\lfloor \frac{l-1}{2} \rfloor} \binom{l}{2m+1}\binom{n-2m-1}{l-1}.$$

Note that $E(n,1) = O(n,1) = T(n,1) = 1$, and $E(l,l) = O(l,l) = T(1,l) = l$, so we next consider $n > 2$ and $1 < l < n$. For these values of n and l, we interpret $E(n,l)$ as the number of ways, from n people with l potential *officers*, to choose a first subset consisting of an even number of officers, and a second disjoint subset of size $l-1$ (the *committee*). Similarly, we interpret $O(n,l)$ as the number of ways to choose an odd number of officers and a disjoint committee of size $l-1$.

Consider $E(n,l)$. Distinguish one of the possible officers, x, and one of the $n-l$ people that is not a potential officer, y. The number of combinations with x in the committee is $E(n-1,l-1)$, as one choice has been removed from the officers and one committee member determined. The number of combinations with y in the committee but with x chosen as neither an officer nor in the committee, is $E(n-2,l-1)$. Two choices were removed overall. One choice of officer was removed, and we need only $l-2$ more for the committee. Finally, with x chosen as an officer or neither x nor y on the committee, there are $O(n-1,l)$ combinations as follows: remove y from the n choices, and choose an odd number of officers and a committee of size $l-1$ as normal. If x has been chosen as an officer, simply remove x and we obtain all combinations with an even number of officers and neither x nor y in the committee. If x has not been chosen as an officer, then add x to the officers; if x was already an element of the committee, remove it and change it to y. This counts all the cases in which x is an officer regardless of the presence of y in the committee.

Hence,

$$E(n,l) = E(n-1,l-1) + E(n-2,l-1) + O(n-1,l). \tag{7}$$

A similar argument shows that

$$O(n,l) = O(n-1,l-1) + O(n-2,l-1) + E(n-1,l).$$

We proceed by fixing l and using strong induction on n, with a base case of $n = l+1$. An easy calculation shows that $E(l+1,l) = O(l+1,l) = l^2$, and also $T(2,l) = l^2$.

Now, suppose that

$$E(p,l) = O(p,l) = T(p-l+1,l)$$

for some integer $p \geq l+1$. Then by (7),

$$E(p+1,l) = E(p,l-1) + E(p-1,l-1) + O(p,l).$$

By the induction hypothesis, we have

$$E(p+1,l) = T(p-l+2,l-1) + T(p-l+1,l-1) + T(p-l+1,l).$$

By the recurrence for T, we have $E(p+1,l) = T(p-l+2,l)$. Further, $O(p+1,l) = T(p-l+2,l)$ by the same argument, and so $E(p+1,l) = O(p+1,l) = T(p-l+2,l)$, as was to be shown. □

This verifies our claim that the final sum of the error terms is 0. Noting that it is easy to verify (1) algebraically for small values of n and k, we have proved the following theorem.

Theorem 3.2. *For $n, k \geq 0$,*

$$\binom{n}{k} = \binom{2n+k+1}{k} + \sum_{j=1}^{k} (-1)^j \frac{n+k+1}{j}\binom{n+k-j}{j-1}\binom{2n+k+1-2j}{k-j}.$$

4. Concluding Remarks

The standard closed form expression [5] for the cross-polytope numbers is

$$T(n,k) = \sum_{m=0}^{n-1} \binom{n-1}{m}\binom{k+m}{n}.$$

In Bodeen et al. [2], a second formula was found, using $2 \times n$ tilings of triangular strips. In notation convenient to ours, their formula is

$$I(n,k) = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2m+1}\binom{n-k+2m+1}{k-1}.$$

It is not hard to show, using standard binomial identities, that $I(n,k)$ is term by term equal to $E(n,k)$ when k is odd and $O(n,k)$ when k is even. It would be of interest to find fundamental relationships between the four formulas.

References

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Department of Mathematics, Kennesaw State University, Marietta, GA 30060
E-mail address: `sedwar77@kennesaw.edu`

Department of Mathematics, Kennesaw State University, Marietta, GA 30060
E-mail address: `wgriff17@kennesaw.edu`