USING MATRICES TO DERIVE IDENTITIES FOR RECURSIVE SEQUENCES

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Abstract. Using matrix methods several identities and binomial summation formulas are obtained for a variety of recursive second and third order sequences. Some well-known summation identities will be extended to identities with negative subscripts.

1. Introduction

Several papers have used matrices to develop (primarily) summation identities for second order sequences and polynomials. The procedure is as follows.

Begin with a seed matrix, squaring in some cases, use the Cayley-Hamilton Theorem on the characteristic equation, completing the square and apply the binomial expansion to obtain summation identities.

The use of matrices to develop binomial summation identities first appeared in *The Fibonacci Quarterly* [7, 8] where several were obtained for the Fibonacci and Lucas numbers. In [7] they began with the first five rows of Pascal's Triangle and in [8] they used the matrices $\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$; $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$; $\begin{bmatrix} L_{2k} & 1 \\ -1 & -1 \end{bmatrix}$; $\begin{bmatrix} F_{2n+2k} & F_{2nk} \\ -F_{2nk} & -F_{2n-2k} \end{bmatrix}$, concluding with matrices for Chebyshev and Fibonacci polynomials, suggesting possibilities for further identities.

Matrices for the Pell and Pell-Lucas polynomial identities were developed in [18] and several binomial summation identities were obtained. If x is set equal to one in these Pell polynomial formulas, several Pell numbers binomial summation identities analogous to those for the Fibonacci numbers in [8] are obtained.

Many other papers have used the matrix method to develop identities but do not address the types of identities emphasized in this paper. For example the interested reader can find the *seed* matrices used for these sequences in the cited references listed in this paper.

Some Pell and Pell-Lucas number identities were developed in [5]. In addition to these, also covered in [3], the modified Pell numbers were considered where it was noted that these numbers were *easily* transferable to Pell-Lucas numbers. The Pell recurrence was generalized to higher dimension and matrices were used in [13] to obtain various relationships but the binomial sums considered in this paper were not considered there. However some forward and backward binomial summation identities were explored in [14].

The matrix method for generating identities for the Jacobsthal numbers can be found, for example, in [3, 15] and for the Jacobsthal-Lucas numbers in [16] but again, the identities obtained are not those considered in this work.

In addition, binomial identities abound in the literature. For example, several involving Fibonacci and Lucas numbers can be found in [17, 21]. Note that the Sloane numbers, as they appear in [19], will be indicated where appropriate throughout this paper.

2. The General kth Order Binomial Identity

Consider a general kth order sequence where $a_0, a_1, \dots, a_{k-1}$ are constants with $a_0 \neq 0$ and for any $n \geq 0$

$$a_{n+k} = \sum_{i=0}^{k-1} p_i a_{n+i}. \tag{2.1}$$

A linear system of equations can be generated from (2.1) by

$$\begin{cases} a_{n+k} = p_{k-1}a_{n+k-1} + p_{k-2}a_{n+k-2} + \cdots + p_1 a_1 + p_0 a_n \\ a_{n+k-1} = a_{n+k-1} \\ \vdots \\ a_{n+1} = a_{n+1}. \end{cases}$$

Let

$$\mathbf{v}_n = \begin{bmatrix} a_{n+k-1} \\ a_{n+k-2} \\ a_{n+k-3} \\ \vdots \\ a_n \end{bmatrix}, \mathbf{S} = \begin{bmatrix} p_{k-1} & p_{k-2} & \dots & p_1 & p_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} a_{k-1} \\ a_{k-2} \\ a_{k-3} \\ \vdots \\ a_0 \end{bmatrix}.$$

It is easy to see the inverse of $\mathbf{S}$ is given by

$$\mathbf{S}^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 \\ \frac{1}{p_0} & -\frac{p_{k-1}}{p_0} & -\frac{p_{k-2}}{p_0} & \dots & -\frac{p_1}{p_0} \end{bmatrix}.$$

It follows by induction that for any integer n

$$\mathbf{v}_n = \mathbf{S}^n \cdot \mathbf{v}_0.$$

Let $\mathbf{R} = \mathbf{S}^m$ for some positive integer m. By the Cayley-Hamilton Theorem $\mathbf{R}$ satisfies its characteristic equation

$$\mathbf{R}^k + b_{k-1}\mathbf{R}^{k-1} + \cdots + b_1\mathbf{R} + b_0\mathbf{I} = \mathbf{0}. \tag{2.2}$$

Suppose that by adding γR^j for some constant γ that the left side of

$$\mathbf{R}^k + b_{k-1}\mathbf{R}^{k-1} + \cdots + b_1\mathbf{R} + b_0\mathbf{I} + \gamma\mathbf{R}^j = \gamma\mathbf{R}^j \tag{2.3}$$

is a binomial expansion. That is, (2.3) differs from a perfect power by a single term and we have *completed the power* so that (2.3) can be written in one of the forms

$$\begin{cases} (\mathbf{R} + \beta\mathbf{I})^k = \gamma R^j, \text{if } j < k \\ (\alpha\mathbf{R} + \beta\mathbf{I})^{k-1} = R^k \text{ or } (\alpha\mathbf{R} + \beta\mathbf{I})^k = \gamma R^k, \text{if } j = k. \end{cases}$$

If this is the case then (2.3) can then be written as one of two types $(\alpha\mathbf{R} + \beta\mathbf{I})^{k-1} = \gamma\mathbf{R}^j$ for $j < k$ or $(\alpha\mathbf{R} + \beta\mathbf{I})^k = \gamma\mathbf{R}^j$ for $j \leq k$ for some constants α, β, γ.

If n is any non-negative integer then the binomial expansions of the two forms above are

$$\sum_{i=0}^{kn} \binom{kn}{i} \alpha^i \beta^{kn-i} \mathbf{S}^{mi} = \gamma^n \mathbf{S}^{mjn} \text{ or } \sum_{i=0}^{(k-1)n} \binom{(k-1)n}{i} \alpha^i \beta^{(k-1)n-i} \mathbf{S}^{mi} = \gamma^n \mathbf{S}^{mjn}.$$

Multiplying each side above by a $\mathbf{S}^K$ for some fixed integer K and then by $\mathbf{v}_0$ results in

$$\sum_{i=0}^{kn}\binom{kn}{i}\alpha^i\beta^{kn-i}\mathbf{S}^{mi+K}\mathbf{v}_0=\gamma^n\mathbf{S}^{mjn+K}\mathbf{v}_0$$

or

$$\sum_{i=0}^{(k-1)n}\binom{(k-1)n}{i}\alpha^i\beta^{(k-1)n-i}\mathbf{S}^{mi+K}\mathbf{v}_0=\gamma^n\mathbf{S}^{mjn+K}\mathbf{v}_0.$$

Which implies for any nonnegative integer n and any integer K that

$$\begin{aligned}&\sum_{i=0}^{kn}\binom{kn}{i}\alpha^i\beta^{kn-i}a_{mi+K}=\gamma^n a_{mjn+K}\text{ or}\\&\sum_{i=0}^{(k-1)n}\binom{(k-1)n}{i}\alpha^i\beta^{(k-1)n-i}a_{mi+K}=\gamma^n a_{mjn+K}.\end{aligned}\tag{2.4}$$

In either case (2.4) yields a binomial identity for every choice of K. We also note that (2.3) may not be unique. For example, $\mathbf{R}^2+4\mathbf{R}+\mathbf{I}=\mathbf{0}$ can be rewritten either as $(\mathbf{R}+2\mathbf{I})^2=3\mathbf{I}$ or $(\mathbf{R}+\mathbf{I})^2=-2\mathbf{I}$, so it may be possible to *complete the power* in any number of different ways thereby yielding various binomial identities using (2.4).

3. Second Order Sequences

Consider a second order linear sequence with initial conditions $a_0=a$, $a_1=b$, and

$$a_{n+2}=pa_{n+1}+qa_n,\ \text{for } n\geq 0.\tag{3.1}$$

Let

$$\mathbf{S}=\begin{bmatrix}p&q\\1&0\end{bmatrix},\mathbf{S}^{-1}=-\frac{1}{q}\begin{bmatrix}0&-q\\-1&p\end{bmatrix},\mathbf{v}_n=\begin{bmatrix}a_{n+1}\\a_n\end{bmatrix},\ \text{and } \mathbf{v}_0=\begin{bmatrix}a_1\\a_0\end{bmatrix},\tag{3.2}$$

so that for any integer n, $\mathbf{S}^n\mathbf{v}_0=\mathbf{v}_n$.

In [24] the matrix S in (3.2) is investigated in some detail and various identities have been obtained but none of those is considered in this paper.

The following proposition shows that it is always possible to find an eigenvalue equation for even powers of $\mathbf{S}$ from (3.2) which allows for the completing of a square in a natural way thus enabling binomial summation identities to be determined.

Proposition 3.1. *For a second order linear recursion, any integer i, and any positive integers m,n we have*

$$\left\{\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\operatorname{tr}(\mathbf{S}^m)^{n-k}a_{m(n+k)+i}=q^{mn}a_i,\\&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}q^{m(n-k)}\operatorname{tr}(\mathbf{S}^m)^k a_{mk+i}=a_{2mn+i},\end{aligned}\right.\quad\textit{for any even } m.$$

$$\left\{\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\operatorname{tr}(\mathbf{S}^m)^{n-k}a_{m(n+k)+i}=q^{mn}a_i,\\&\sum_{k=0}^{n}\binom{n}{k}q^{m(n-k)}\operatorname{tr}(\mathbf{S}^m)^k a_{mk+i}=a_{2mn+i},\end{aligned}\right.\quad\textit{for any odd } m.$$

Proof. Let $\mathbf{R} = \mathbf{S}^m$ for some positive integer m. The characteristic equation of $\mathbf{R}$ satisfies $\lambda^2 - \mathrm{tr}(\mathbf{R})\lambda + \det(\mathbf{R}) = 0$. Now $\det(\mathbf{R}) = \det(\mathbf{S}^m) = (-q)^m$ and so $\mathbf{R}$ satisfies $\mathbf{R}(\mathbf{R} - \mathrm{tr}(\mathbf{R})\mathbf{I}) = -(-1)^m q^m \mathbf{I}$. Thus we have $\mathbf{R}^n(\mathbf{R} - \mathrm{tr}(\mathbf{R})\mathbf{I})^n = (-1)^n(-1)^{mn}q^{mn}\mathbf{I}$ and so

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathrm{tr}(\mathbf{R})^{n-k}\mathbf{R}^{n+k}\mathbf{S}^i = (-1)^n(-1)^{mn}q^{mn}\mathbf{S}^i.$$

So,

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}\mathbf{S}^{m(n+k)+i}\mathbf{v}_0 = (-1)^n(-1)^{mn}q^{mn}\mathbf{S}^i\mathbf{v}_0$$

and thus,

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}a_{m(n+k)+i} = (-1)^n(-1)^{mn}q^{mn}a_i.$$

Note that if m is odd then $(-1)^{mn} = (-1)^n$ so $(-1)^{2n} = 1$. If m is even then $(-1)^{mn} = 1$ and $(-1)^{n-k} = (-1)^{n+k}$ so there is a common factor of $(-1)^n$ in the even case. This proves two of the identities. To prove the remaining two identities, note that $\mathbf{R}$ also satisfies $\mathbf{R}^2 = \mathrm{tr}(\mathbf{R})\mathbf{R} - (-q)^m\mathbf{I}$ thus, $(\mathrm{tr}(\mathbf{R})\mathbf{R} - (-q)^m\mathbf{I})^n = \mathbf{R}^{2n}$ and so

$$\sum_{k=0}^{n} (-1)^{(m+1)(n-k)}\binom{n}{k}q^{m(n-k)}\mathrm{tr}(\mathbf{R})^k\mathbf{R}^k\mathbf{S}^i = \mathbf{R}^{2n}\mathbf{S}^i.$$

So,

$$\sum_{k=0}^{n} (-1)^{(m+1)(n-k)}\binom{n}{k}q^{m(n-k)}\mathrm{tr}(\mathbf{S}^m)^k\mathbf{S}^{mk+i}\mathbf{v}_0 = \mathbf{S}^{2mn+i}\mathbf{v}_0$$

and thus,

$$\sum_{k=0}^{n} (-1)^{(m+1)(n-k)}\binom{n}{k}q^{m(n-k)}\mathrm{tr}(\mathbf{S}^m)^k a_{mk+i} = a_{2mn+i}.$$

$\square$

Note that 3.1 is valid for any choice of integer i and so holds true for negative subscripts as well. A more readable form for negative subscripts is the following.

Proposition 3.2. *For a second order linear recursion, any integer i, and any positive integers m, n we have*

$$\begin{cases} \displaystyle\sum_{k=0}^{n} (-1)^k\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}q^{mk}a_{-m(n+k)-i} = a_{-i}, \\ \displaystyle\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^k a_{-mk-i} = q^{mn}a_{-2mn-i}, \end{cases} \quad \textit{for any even } m.$$

$$\begin{cases} \displaystyle\sum_{k=0}^{n} \binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}q^{mk}a_{-m(n+k)-i} = a_{-i}, \\ \displaystyle\sum_{k=0}^{n} (-1)^k\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^k a_{-mk-i} = q^{mn}a_{-2mn-i}, \end{cases} \quad \textit{for any odd } m.$$

Proof. Since $\mathbf{R}^2 - \mathrm{tr}(\mathbf{R})\mathbf{R} + (-q)^m\mathbf{I} = \mathbf{0}$ then $\mathbf{I} - \mathrm{tr}(\mathbf{R})\mathbf{R}^{-1} + (-q)^m\mathbf{R}^{-2} = \mathbf{0}$ so $\mathbf{I} = (\mathrm{tr}(\mathbf{R})\mathbf{I} - (-q)^m\mathbf{R}^{-1})\mathbf{R}^{-1}$ and $(\mathrm{tr}(\mathbf{R})\mathbf{R}^{-1} - \mathbf{I}) = (-q)^m\mathbf{R}^{-2}$. Hence, it follows that $\mathbf{S}^{-i} = (\mathrm{tr}(\mathbf{R})\mathbf{I} - (-q)^m\mathbf{R}^{-1})^n\mathbf{R}^{-n}\mathbf{S}^{-i}$ and $(\mathrm{tr}(\mathbf{R})\mathbf{R}^{-1} - \mathbf{I})^n\mathbf{S}^{-i} = (-q)^{mn}\mathbf{R}^{-2n}\mathbf{S}^{-i}$. Thus,

$$\sum_{k=0}^{n}(-1)^{(m+1)k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}q^{mk}\mathbf{S}^{-m(k+n)-i} = \mathbf{S}^{-i}$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{k}\mathbf{S}^{-mk-i} = (-1)^{mn}q^{mn}\mathbf{S}^{-2mn-i}$$

and the proposition now follows. □

Corollary 3.3. *If $m = 1$ then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}p^{n-k}a_{n+k+i} = q^n a_i$$
$$\sum_{k=0}^{n}\binom{n}{k}q^{n-k}p^k a_{k+i} = a_{2n+i}$$
$$\sum_{k=0}^{n}\binom{n}{k}p^{n-k}q^k a_{-n-k-i} = a_{-i}$$
$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}p^k a_{-k-i} = q^n a_{-2mn-i}$$

for a second order linear recursion.

Corollary 3.4. *If $a_0 = 0$ then*

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\mathrm{tr}(\mathbf{S}^m)^{n-k}a_{m(n+k)} = 0$$
$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}p^{n-k}a_{n+k} = 0$$

for a second order linear recursion.

Corollary 3.5. *For the Fibonacci numbers we have*

$$\begin{cases} \displaystyle\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}L_m^{n-k}F_{m(n+k)+i} = F_i, & \\ & \text{for any even } m. \\ \displaystyle\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}L_m^{k}F_{mk+i} = F_{2mn+i}, & \end{cases}$$

$$\begin{cases} \displaystyle\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}L_m^{n-k}F_{m(n+k)+i} = F_i, & \\ & \text{for any odd } m. \\ \displaystyle\sum_{k=0}^{n}\binom{n}{k}L_m^{k}a_{mk+i} = F_{2mn+i}, & \end{cases}$$

where L_m is the mth Lucas number.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_{n+k+i} = F_i,$$
$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}F_{n+k} = 0,$$
$$\sum_{k=0}^{n}\binom{n}{k}F_{k+i} = F_{2n+i},$$
$$\sum_{k=0}^{n}\binom{n}{k}F_{k} = F_{2n}.$$

Proof. For the Fibonacci numbers we have for any m that $\mathrm{tr}(\mathbf{S}^m) = F_{m+1} + F_{m-1} = L_m$ and so the corollary follows. □

The next proposition shows that for $\mathbf{S}^{2m}$, i.e., even powers of $\mathbf{S}$ that $-2q^m$ occurs naturally as a term in the trace of $\mathbf{S}^{2m}$ and so there is a natural way to complete the square in the characteristic polynomial of $\mathbf{S}^m$.

Proposition 3.6. *If $\boldsymbol{S}$ is the matrix of* (3.2) *then the characteristic equation of $\mathbf{S}^{2m}$ satisfies*

$$\lambda^2 + (-\mathrm{tr}(\boldsymbol{S}^m)^2 - 2q^m)\lambda + q^{2m} = 0, \textit{ if } m \textit{ is odd.}$$
$$\lambda^2 + (-\mathrm{tr}(\boldsymbol{S}^m)^2 + 2q^m)\lambda + q^{2m} = 0, \textit{ if } m \textit{ is even.}$$

Proof. For any even positive integer $2m$ we have that the characteristic equation of $\mathbf{S}^{2m}$ satisfies $\lambda^2 - \mathrm{tr}(\mathbf{S}^{2m})\lambda + q^{2m} = 0$. Let $\gamma = \sqrt{\lambda}$ so that $\det(\mathbf{S}^{2m} - \lambda\mathbf{I}) = \det(\mathbf{S}^m - \gamma\mathbf{I})\det(\mathbf{S}^m + \gamma\mathbf{I})$. If m is odd then

$$\begin{aligned} 0 = \det(\mathbf{S}^m - \gamma\mathbf{I})\det(\mathbf{S}^m + \gamma\mathbf{I}) &= (\gamma^2 - \mathrm{tr}(\mathbf{S}^m)\gamma - q^m)(\gamma^2 + \mathrm{tr}(\mathbf{S}^m)\gamma - q^m) \\ &= \gamma^4 + (-\mathrm{tr}(\mathbf{S}^m)^2 - 2q^m)\gamma^2 + q^{2m}. \end{aligned}$$

If m is even then

$$\begin{aligned} 0 = \det(\mathbf{S}^m - \gamma\mathbf{I})\det(\mathbf{S}^m + \gamma\mathbf{I}) &= (\gamma^2 - \mathrm{tr}(\mathbf{S}^m)\gamma + q^m)(\gamma^2 + \mathrm{tr}(\mathbf{S}^m)\gamma + q^m) \\ &= \gamma^4 + (-\mathrm{tr}(\mathbf{S}^m)^2 + 2q^m)\gamma^2 + q^{2m}. \end{aligned}$$

□

It follows that there are at least two ways to complete the square using the middle term for any even power of the matrix $\mathbf{S}$ generated by a second order recursion relation by using the q^{2m} constant term and the $2q^m$ coefficient. That is by the Cayley-Hamilton Theorem: $\mathbf{S}^{4m} + (-\mathrm{tr}(\mathbf{S}^m)^2 \pm 2q^m)\mathbf{S}^{2m} + q^{2m}\mathbf{I} = \mathbf{0}$ implies

$$(\mathbf{S}^{2m} - q^m\mathbf{I})^2 = \mathrm{tr}(\mathbf{S}^m)^2\mathbf{S}^{2m} \text{ and } (\mathbf{S}^{2m} + q^m\mathbf{I})^2 = (\mathrm{tr}(\mathbf{S}^m)^2 + 4q^m)\mathbf{S}^{2m} \text{ if } m \text{ is odd,}$$
$$(\mathbf{S}^{2m} + q^m\mathbf{I})^2 = \mathrm{tr}(\mathbf{S}^m)^2\mathbf{S}^{2m} \text{ and } (\mathbf{S}^{2m} - q^m\mathbf{I})^2 = (\mathrm{tr}(\mathbf{S}^m)^2 - 4q^m)\mathbf{S}^{2m} \text{ if } m \text{ is even.}$$

In particular for $\mathbf{S}^2$ we have $(\mathbf{S}^2 - q\mathbf{I})^2 = p^2\mathbf{S}^2$ and $(\mathbf{S}^2 + q\mathbf{I})^2 = (p^2 + 4q)\mathbf{S}^2$.

Similarly $\mathbf{S}^{4m} + (-\mathrm{tr}(\mathbf{S}^m)^2 \pm 2q^m)\mathbf{S}^{2m} + q^{2m}\mathbf{I} = \mathbf{0}$ implies $\mathbf{I} + (-\mathrm{tr}(\mathbf{S}^m)^2 \pm 2q^m)\mathbf{S}^{-2m} + q^{2m}\mathbf{S}^{-4m} = \mathbf{0}$ so

$$(q\mathbf{S}^{-2} - \mathbf{I})^2 = p^2\mathbf{S}^{-2} \text{ and } (q\mathbf{S}^{-2} + \mathbf{I})^2 = (p^2 + 4q)\mathbf{S}^{-2}.$$

Proposition 3.7. *For any whole number i the following identities hold for a second order recursion $a_{n+2} = pa_{n=1} + qa_n, a_0 = a, a_1 = b$*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^{2n-k} a_{2k+i} = p^{2n} a_{2n+i},$$
$$\sum_{k=0}^{2n} \binom{2n}{k} q^{2n-k} a_{2k+i} = (p^2 + 4q)^n a_{2n+i}.$$

Proof. We have $(\mathbf{S}^2 - q\mathbf{I})^{2n} = (p^2\mathbf{S}^2)^n$ and $(\mathbf{S}^2 + q\mathbf{I})^{2n} = ((p^2 + 4q)\mathbf{S}^2)^n$ so by the binomial theorem we have

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k} = p^{2n}\mathbf{S}^{2n} \text{ and } \sum_{k=0}^{2n} \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k} = (p^2 + 4q)^n \mathbf{S}^{2n}.$$

Multiplying all sides above by $\mathbf{S}^i$ for any whole number i we have

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k+i} = p^{2n}\mathbf{S}^{2n+i} \text{ and } \sum_{k=0}^{2n} \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k+i} = (p^2 + 4q)^n \mathbf{S}^{2n+i}.$$

Hence,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k+i} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = p^{2n}\mathbf{S}^{2n+i} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} q^{2n-k} \mathbf{S}^{2k+i} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = (p^2 + 4q)^n \mathbf{S}^{2n+i} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}.$$

From (3.2) it follows that for any whole number i

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^{2n-k} a_{2k+i} = p^{2n} a_{2n+i}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} q^{2n-k} a_{2k+i} = (p^2 + 4q)^n a_{2n+i}.$$

□

For negative subscripts we have the following proposition.

Proposition 3.8. *For any whole number i the following identities hold for a second order recursion $a_n = \frac{1}{q} a_{n+2} - \frac{p}{q} a_{n+1}, a_0 = a, a_1 = b$*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} q^k a_{-2k-i} = p^{2n} a_{-2n-i},$$
$$\sum_{k=0}^{2n} \binom{2n}{k} q^k a_{-2k-i} = (p^2 + 4q)^n a_{-2n-i}.$$

Proof. We have $(q\mathbf{S}^{-2}-\mathbf{I})^{2n}=(p^2\mathbf{S}^{-2})^n$ and $(q\mathbf{S}^{-2}+\mathbf{I})^{2n}=((p^2+4q)\mathbf{S}^{-2})^n$ so again by the binomial theorem and by multiplying all sides $(\mathbf{S}^{-1})^i$ for any positive integer i we have

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}q^k\mathbf{S}^{-2k-i}\begin{bmatrix} a_0 \\ a_{-1}\end{bmatrix}=p^{2n}\mathbf{S}^{-2n-i}\begin{bmatrix} a_0 \\ a_{-1}\end{bmatrix}$$

and

$$\sum_{k=0}^{2n}\binom{2n}{k}q^k\mathbf{S}^{-2k+i}\begin{bmatrix} a_0 \\ a_{-1}\end{bmatrix}=(p^2+4q)^n\mathbf{S}^{-2n+i}\begin{bmatrix} a_0 \\ a_{-1}\end{bmatrix}.$$

Thus it follows that for any whole number i

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}q^k a_{-2k-i}=p^{2n}a_{-2n-i}$$

and

$$\sum_{k=0}^{2n}\binom{2n}{k}q^k a_{-2k-i}=(p^2+4q)^n a_{-2n-i}.$$

$\square$

By way of examples we present the following binomial sums.

3.1. **Fibonacci Sequence.** $F_n,(p=1,q=1)$. Here $\mathbf{S}=\begin{bmatrix} 1 & 1 \\ 1 & 0\end{bmatrix}$, $\mathbf{S}^2=\begin{bmatrix} 2 & 1 \\ 1 & 1\end{bmatrix}$, and $\mathbf{S}^{-2}=\begin{bmatrix} 1 & -1 \\ -1 & 2\end{bmatrix}$. By Proposition 3.8 we have

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}F_{-2k-i}=F_{-2n-i}\text{ and }\sum_{k=0}^{2n}\binom{2n}{k}F_{-2k-i}=5^nF_{-2n-i}.$$

3.2. **Pell Sequence.** $P_n,(p=2,q=1)$, [1, 3]. Here $\mathbf{S}=\begin{bmatrix} 2 & 1 \\ 1 & 0\end{bmatrix}$, $\mathbf{S}^2=\begin{bmatrix} 5 & 2 \\ 2 & 1\end{bmatrix}$, and $\mathbf{S}^{-2}=\begin{bmatrix} 1 & -2 \\ -2 & 5\end{bmatrix}$. Using Proposition 3.7 yields the following identities that vary slightly with those obtained from [18]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}P_{2k+i}=4^nP_{2n+i}\text{ and }\sum_{k=0}^{2n}\binom{2n}{k}P_{2k+i}=8^nP_{2n+i}.$$

By Proposition 3.8 for negative subscripts we have

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}P_{-2k-i}=4^nP_{-2n-i}\text{ and }\sum_{k=0}^{2n}\binom{2n}{k}P_{-2k-i}=8^nP_{-2n-i}.$$

3.3. **Jacobsthal Sequence.** $J_n, (p = 1, q = 2)$, [4]. Here $\mathbf{S} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{S}^2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$, and $\mathbf{S}^{-2} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$. Using Proposition 3.7 yields the following identities

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k}J_{2k+i} = J_{2n+i} \text{ and } \sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k}J_{2k+i} = 9^n J_{2n+i}.$$

By Proposition 3.8 for negative subscripts we have

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{k}J_{-2k-i} = J_{-2n-i} \text{ and } \sum_{k=0}^{2n}\binom{2n}{k}2^{k}J_{-2k-i} = 9^n J_{-2n-i}.$$

Next we consider polynomials associated with the classical Chebyshev polynomials described by the recurrence $A_{n+2}(x) = 2xA_{n+1}(x) - A_n(x)$.

In [12] associated Chebyshev polynomials of the second kind $A_n = U_n$ with $U_0 = 1$ and $U_1 = 2x$; were investigated and in [10] associated Chebyshev polynomials of the first kind $A_n = T_n$ with $T_0 = 2$ and $T_1 = 2x$ were analogously considered.

Furthermore Horadam described Fermat polynomials of the first kind, $\mathfrak{F}_n(x)$, and the second kind, $\mathfrak{f}_n(x)$, in [9] as follows. Both satisfy the recurrence $\mathfrak{F}_{n+2}(x) = 3x\mathfrak{F}_{n+1}(x) - 2\mathfrak{F}_n(x)$ with initial conditions, respectively as $\mathfrak{F}_0(x) = 0, \mathfrak{F}_1(x) = 1$; and $\mathfrak{f}_0(x) = 2, \mathfrak{f}_1(x) = 3x$. He called the second kind Fermat-Lucas polynomials.

The Fermat numbers (not to be confused with the classical Fermat numbers F_n) are obtained by letting $x = 1$. He called $\mathfrak{F}(1) = \phi_n$ yielding the sequence $\{\phi_n\} = \{0, 1, 3, 7, 15, \ldots\}$ with Sloane number A000225 sometimes called Mersenne numbers ($2^n - 1$ as opposed to the classical Mersenne numbers where n is prime) and $\mathfrak{f}_n(1) = \theta_n$ yielding the sequence $\{\theta_n\} = \{2, 3, 5, 9, 17, \ldots\} = \{2^n + 1\}$ with Sloane number A000051, sometimes referred to as the Pisot sequence $L(2, 3)$. Both sequences satisfy the recurrence $A_{n+2} = 3A_{n+1} - 2A_n$.

Further investigation of these and other polynomial sequences leading to the second order sequential numbers considered in this paper has also been explored in [11].

3.4. **Chebyshev Numbers.** $x = 1, A_{n+2} = 2A_{n+1} - A_n$.

Setting $x = 1$ in the Chebyshev polynomials yields the common recurrence

$$A_{n+2} = 2A_{n+1} - A_n$$

for Chebyshev numbers. In this case it follows by induction that $\mathbf{S}^n = \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix}$ and that the characteristic equation for any power of $\mathbf{S}$ is

$$\det(\mathbf{S}^n - \lambda\mathbf{I}) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

For $n = 1$ this implies $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\mathbf{S}^k = \mathbf{0}$ so that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}a_{k+1} = 0 \text{ and } \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}a_k = 0.$$

At first glance this relationship might seem less interesting than the previous ones. However when the arbitrary initial conditions, a_0 and a_1 are assigned values $a_0 = 0$ and $a_1 = 1$, the Chebyshev numbers of the first kind become $\{1, 1, 1, 1, \ldots, 1, \ldots\}$ while $a_0 = 1$ and $a_1 = 2$, the Chebyshev numbers of the first kind become $\{1, 2, 3, 4, \ldots, n, \ldots\}$. These initial conditions yield the identities

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k} = 0 \text{ and } \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}k = 0$$

which normally require a proof by induction.

3.5. **Fermat Numbers.** These Fermat numbers are not the classical ones. For these Fermat numbers, $p = 3$ and $q = -2$. Using Proposition 3.7 we get the following identities:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(-2)^{2n-k}a_{2k+i} = \sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k}a_{2k+i} = 9^n a_{2n+i}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}(-2)^{2n-k}a_{2k+i} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k}a_{2k+i} = a_{2n+i},$$

when $a_j = \phi_j = 2^j - 1, a_j = \theta_j = 2^j + 1$ then we have

$$\sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k}(2^{2k+i} - 1) = 9^n(2_{2n+i} - 1) \text{ and } \sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k}(2^{2k+i} + 1) = 9^n(2_{2n+i} + 1)$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k}(2^{2k+i} - 1) = 2^{2n+i} - 1 \text{ and } \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k}(2^{2k+i} + 1) = 2^{2n+i} + 1.$$

This results in the identities: $\sum_{k=0}^{2n}\binom{2n}{k}2^{2n+k+i} = 9^n 2^{2n+i}, \sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k} = 9^n$ and $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n+k+i} = 2^{2n+i}, \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k} = 1$. This reduces to

$$\sum_{k=0}^{2n}\binom{2n}{k}2^k = 9^n, \sum_{k=0}^{2n}\binom{2n}{k}2^{2n-k} = 9^n,$$

and

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^k = 1, \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-k} = 1.$$

3.6. **Sloane A001109.** $a_{n+2} = 6a_{n+1} - a_n, a_0 = 0, a_1 = 1, p = 6, q = 1$ (a_n^2 is a Triangular number).

Here $\mathbf{S}^2 - 6\mathbf{S} + \mathbf{I} = \mathbf{0}$ so $(3\mathbf{S} - \mathbf{I})^2 = 8\mathbf{S}^2, (\mathbf{S} - 3\mathbf{I})^2 = 8\mathbf{I}, (\mathbf{S} - \mathbf{I})^2 = 4\mathbf{S}, (\mathbf{S} + \mathbf{I})^2 = 8\mathbf{S}$ and for any integer $i \geq 0$

$$\sum_{k=0}^{2n}(-1)^k 3^k a_{k+i} = 8^n a_{2n+i} \text{ and } \sum_{k=0}^{2n}(-1)^k 3^{2n-k} a_{k+i} = 8^n a_{n+i}.$$

4. Third Order Sequences

The matrix technique introduced in [8] has been extended to third order sequences for specific sequences in [20, 22] and some identities were provided, with suggestions for further development. The general case analogous to (3.2) was considered in [23] where some summation identities not involving binomial coefficients were obtained for some specific cases. Additional algorithmic techniques were provided in [2] but the matrix method was not employed.

Investigating several powers of the basic matrix using MAPLE suggests that a useful eigenvalue equation analogous to that determined in Proposition 3.6 for the second order case may not be obtainable. However some of the more well-known cases yield summation identities. These are illustrated here.

We consider third order linear sequences with initial conditions $a_0 = a, a_1 = b, a_2 = c$ and

$$a_{n+3} = pa_{n+2} + qa_{n+1} + ra_n, \tag{4.1}$$

for $n \geq 0$. Let

$$\mathbf{S} = \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ so that } \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}. \tag{4.2}$$

4.1. **The Tribonacci Sequence.** Here $p = 1, q = 1, r = 1$. $\mathbf{S}^2$ satisfies its characteristic equation $\lambda^3 - 3\lambda^2 - \lambda - 1 = 0$ so we have $\mathbf{S}^6 - 3\mathbf{S}^4 - \mathbf{S}^2 - \mathbf{I} = \mathbf{0}$ so $(\mathbf{S}^2 - \mathbf{I})^3 = 4\mathbf{S}^2$ and so

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} \mathbf{S}^{2k+i} = 4^n \mathbf{S}^{2n+i}.$$

Thus for any $i \geq 0$

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} a_{2k+i} = 4^n a_{2n+i}. \tag{4.3}$$

Using MAPLE to investigate lower powers of $\mathbf{S}$ are fruitless until $\mathbf{S}^8$ where $\mathbf{S}^8$ satisfies the equation $\lambda^3 - 131\lambda^2 + 3\lambda - 1 = 0$ and so $\mathbf{S}^{24} - 131\mathbf{S}^{16} + 3\mathbf{S}^8 - \mathbf{I} = \mathbf{0}$ we have $(\mathbf{S}^8 - \mathbf{I})^3 = 128\mathbf{S}^{16}$ and so

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} \mathbf{S}^{8k+i} = 128^n \mathbf{S}^{16n+i}.$$

So for any positive integer $i \geq 0$

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} a_{8k+i} = (128)^n a_{16n+i}. \tag{4.4}$$

4.2. **The Perrin-Padovan Sequences** [25, 26]**.** Here $p = 0, q = 1, r = 1$ in (4.1) and $\mathbf{S}^3$ satisfies its characteristic equation $\lambda^3 - 3\lambda^2 + 2\lambda - 1 = 0$. So $\mathbf{S}^9 - 3\mathbf{S}^6 + 2\mathbf{S}^3 - \mathbf{I} = \mathbf{0}$ and so $(\mathbf{S}^3 - \mathbf{I})^3 = \mathbf{S}^3$. Thus,

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} \mathbf{S}^{3k+i} = \mathbf{S}^{3n+i}$$

and so for any positive integer $i \geq 0$

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} a_{3k+i} = a_{3n+i}. \tag{4.5}$$

4.3. **The Narayana's Cows Sequence.** Here $p = 1, q = 0, r = 1$. $\mathbf{S}^3$ satisfies the equation $\lambda^3 - 4\lambda^2 + 3\lambda - 1 = 0$. So $\mathbf{S}^9 - 4\mathbf{S}^6 + 3\mathbf{S}^3 - \mathbf{I} = \mathbf{0}$ and $(\mathbf{S}^3 - \mathbf{I})^3 = \mathbf{S}^6$ and so for any positive integer $i \geq 0$ that

$$\sum_{k=0}^{3n} (-1)^{3n-k} \binom{3n}{k} a_{3k+i} = a_{6n+i}. \tag{4.6}$$

An interesting presentation on the Narayana–Cows Problem and the resulting sequence $\{2, 3, 4, 6, 9, \ldots\}$ with Sloane number A000930 can be found in [6].

4.4. **Sloane A0002478 and Sloane A108122.** Here $p = 1, q = 2$, and $r = 1$ so that $a_{n+3} = a_{n+2} + 2a_{n+1} + a_n$. $\mathbf{S}$ satisfies $\mathbf{S}^3 = (\mathbf{S} + \mathbf{I})^2$ and so

$$\sum_{k=0}^{2n} \binom{2n}{k} \mathbf{S}^{k+i} = \mathbf{S}^{3n+i}$$

and thus for any integer $i \geq 0$

$$\sum_{k=0}^{2n} \binom{2n}{k} a_{k+i} = a_{3n+i}.$$

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